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Abstract

Nonlinear time series models, especially those with regime-switching and GARCH errors, have become increasingly popular in the economics and finance literature. However, much of the research has concentrated on the empirical applications of various models, with little theoretical or statistical analysis associated with the structure of the processes or the associated asymptotic theory. In this paper we derive necessary and sufficient conditions for strict stationarity and ergodicity of three different specifications of the first-order STAR-GARCH model, and sufficient conditions for the existence of moments. This is important, among others, to establish the conditions under which the traditional LM linearity tests based on Taylor expansions are valid. Finally, we provide sufficient conditions for consistency and asymptotic normality of the Quasi-Maximum Likelihood Estimator.
KEYWORDS: Nonlinear time series, regime-switching, STAR, GARCH, log-moment, moment conditions, asymptotic theory.

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1 Introduction

Recent years have witnessed a vast development of nonlinear techniques for modelling the conditional mean and conditional variance of economic and financial time series. In the vast array of new technical developments for conditional mean models, the Smooth Transition AutoRegressive (STAR) specification, proposed by Chan and Tong (1986) and developed by Luukkonen, Saikkonen and Teräsvirta (1988) and Teräsvirta (1994), has found a number of successful applications (see van Dijk, Teräsvirta and Franses (2002) for a recent review). The term "smooth transition" in its present meaning first appeared in Bacon and Watts (1971). They presented their smooth transition specification as a model of two intersecting lines with an abrupt change from one linear regression to another at an unknown change-point. Goldfeld and Quandt (1972, pp. 263-264) generalized the so-called two-regime switching regression model using the same idea. In the time series literature, the STAR model is a natural generalization of the Self-Exciting Threshold Autoregressive (SETAR) models pioneered by Tong (1978) and Tong and Lim (1980) (see also Tong (1990)).

In terms of the conditional variance, Engle's (1982) Autoregressive Conditional Heteroskedasticity (ARCH) model and Bollerslev's (1986) Generalised ARCH (GARCH) model are the most popular specifications for capturing time-varying symmetric volatility in financial and economic time series data.

Despite their popularity, the structural and statistical properties of these models were not fully established until recently. Chan and Tong (1986) derived the sufficient conditions for strict stationarity and geometric ergodicity of a two-regime STAR model, where the transition function is given by the cumulative Gaussian distribution. Consistency and asymptotic normality of the nonlinear least squares estimator are given under the assumption that the errors are homoskedastic and independent. Although several papers have been published in the literature with general conditions for strict stationarity and ergodicity of nonlinear time series models, especially threshold-type models, few attempts have been made to comprehend the dynamics of more general smooth transition processes. In general, only very restrictive sufficient conditions are provided (see Cline and Pu (1999a, 1999b) and Ferrante, Fonseca and Vidoni (2003), among many others). More recently, Mira and Escribano (2000) derived new conditions for consistency and asymptotic normality of the nonlinear least squares estimator. However, estimation of the conditional variance was not considered in these papers.

Significant efforts have been made to fully understand the properties of univariate and multivariate GARCH models. Nelson (1990) derived the necessary and sufficient log-moment condition for stationarity and ergodicity of the GARCH(1,1) model. This condition was extended to higher-order models by Bougerol and Picard (1992). Weak stationarity and the existence of fourth moments of a family of power GARCH models have been investigated in He and Teräsvirta (1999a,b), while Ling and McAleer (2002a,b) derived the necessary and sufficient conditions for the existence of all moments for these models.

Concerning the estimation of parameters for GARCH models, Lee and Hansen (1994) and Lumsdaine (1996) proved that the local Quasi-Maximum Likelihood Estimator (QMLE) was consistent and asymptotic normal under strong conditions. Jeantheau (1998) established the consistency results of estimators for multivariate GARCH models. His proofs of consistency did not assume a particular functional form for the conditional mean, but assumed a log-moment condition and some regularity conditions for purposes of identification. More recently, Ling and McAleer (2003) proposed the vector ARMA-GARCH model and proved the consistency of the global QMLE under only the second-order moment condition. They also proved the asymptotic normality of the global (local) QMLE under the sixth-order (fourth-order) moment condition. Comte and Lieberman (2003) studied the asymptotic properties of the QMLE for the BEKK model of Engle and Kroner (1995). Berkes, Horváth and Kokoszka (2003) proved the consistency and asymptotic normality of the QMLE of the parameters of the GARCH(p,q) model under second- and fourth-order moment conditions, respectively. Boussama (2000), McAleer, Chan and Marinova (2002), and Francq and Zakoïan (2004) also considered the properties of the QMLE under different specifications of the symmetric and asymmetric GARCH(p,q) model.

However, most of the theoretical results on GARCH models have assumed a constant or linear conditional mean (see McAleer (2005) for further details). It has not yet been established whether these results would also hold if the conditional mean were nonlinear. Chan and McAleer (2002) combined the general STAR model with GARCH(p,q) errors, but their results were derived under the assumption that the conditional mean parameters were known.

This paper extends existing results in the literature in several respects. The necessary and sufficient conditions for strict stationarity and geometric ergodicity of a general class of STAR models with GARCH(1,1) errors are established. STAR models with more than two regimes are also considered. Conditions for the existence of moments are also examined. Finally, consistency and asymptotic normality of the QMLE are derived under weak conditions.

The structural and statistical properties developed in this paper can also be used to derive the distributions associated with various test statistics proposed in the nonlinear time series literature. These properties provide the foundation for developing more complete tests for important economic and financial hypotheses. For instance, the correlation between prices over time is often used as a test for the weak form of the Efficient Market Hypothesis (EMH), which assumes that prices follow a linear process. However, if prices follow a nonlinear process, such as a STAR-type process, the correlation between prices over time may appear insignificant in finite samples. Thus, formal tests of nonlinear dependence would also provide an important diagnostic for testing the EMH.

The plan of the paper is as follows: Section 2 provides a description of the models considered in the paper. Stationarity, ergodicity and the existence of moments are discussed in Section 3. The asymptotic properties of the QMLE are considered in Section 4. Finally, Section 5 gives some concluding remarks. All technical proofs are given in the Appendix.

2 Model Specification

In this paper we consider three different classes of STAR-GARCH models. The first specification is an additive logistic STAR model with multiple regimes in the conditional mean and GARCH errors. This model nests the SETAR-GARCH process of Li and Lam (1995). A similar specification with Gaussian errors was proposed in Suarez-Fariñas, Pedreira and Medeiros (2004) and Medeiros and Veiga (2000, 2005) . The second specification is a restricted form of the multiple-regime logistic STAR model with GARCH errors. This particular functional form with homoskedastic errors was discussed in van Dijk, Teräsvirta and Franses (2002). Finally, the third specification is the Exponential STAR-GARCH (ESTAR-GARCH) model, of which the Exponential STAR (ESTAR) (Teräsvirta 1994) model is a special case.

DEFINITION 1. *The* $\mathbb{R}$*-valued process* $\{y_t, t \in \mathbb{Z}\}$ *follows a first-order autoregressive model with time-varying coefficients and GARCH(1,1) errors if*

$$y_t = \lambda_{0,t-1} + \lambda_{1,t-1} y_{t-1} + \varepsilon_t, \tag{1}$$

$$\varepsilon_t = \eta_t \sqrt{h_t}, \text{ and} \tag{2}$$

$$h_t = \omega + \alpha \varepsilon_{t-1}^2 + \beta h_{t-1}, \tag{3}$$

where $\{\eta_t\}$ *is a sequence of independently and identically distributed zero mean and unit variance random variables,* $\eta_t \sim \mathsf{IID}(0,1)$, $\lambda_{0,t-1} = f_0(y_{t-1};\boldsymbol{\lambda})$, $\lambda_{1,t-1} = f_1(y_{t-1};\boldsymbol{\lambda})$, *and* $f_i(y_{t-1};\boldsymbol{\lambda})$, $i = 0, 1$, *is a nonlinear function of* y_{t-1} *indexed by the vector of parameters* $\boldsymbol{\lambda}$.

It is clear that the model defined by equations (1)–(3) is similar to the functional coefficient autoregressive model proposed by Chen and Tsay (1993).

Depending on the choice of the functions $f_0(y_{t-1};\boldsymbol{\lambda})$ and $f_1(y_{t-1};\boldsymbol{\lambda})$, different specifications of the STAR model can be derived. The following cases are considered:

1. The Multiple Regime Logistic STAR(1)-GARCH(1,1) (or MRLSTAR(1)-GARCH(1,1)) model:

$$f_0(y_{t-1};\boldsymbol{\lambda}) = \theta_0 + \sum_{i=1}^{m} \theta_i G(y_{t-1};\gamma_i, c_i), \text{ and} \tag{4}$$

$$f_1(y_{t-1};\boldsymbol{\lambda}) = \phi_0 + \sum_{i=1}^{m} \phi_i G(y_{t-1};\gamma_i, c_i), \tag{5}$$

where

$$G\left(y_{t-1};\gamma_i, c_i\right) = \frac{1}{1 + e^{-\gamma_i (y_{t-1} - c_i)}}, \tag{6}$$

with $\boldsymbol{\lambda} = (\theta_0, \theta_1, \phi_{01}, \ldots, \phi_{0m}, \phi_{11}, \ldots, \phi_{1m}, \gamma_1, \ldots, \gamma_m, c_1, \ldots, c_m)'$.

2. The Generalized STAR(1)-GARCH(1,1) (or GSTAR(1)-GARCH(1,1)) model:

$$f_0(y_{t-1};\boldsymbol{\lambda}) = \theta_0 + \theta_1 G(y_{t-1};\gamma,\mathbf{c}), \text{ and} \tag{7}$$

$$f_1(y_{t-1};\boldsymbol{\lambda}) = \phi_0 + \phi_1 G(y_{t-1};\gamma,\mathbf{c}), \tag{8}$$

where

$$G\left(y_{t-1};\gamma,c\right) = \frac{1}{1+e^{-\gamma\left[\prod_{i=1}^{m}(y_{t-1}-c_i)\right]}}, \tag{9}$$

with $\mathbf{c} = (c_1,\ldots,c_m)'$ and $\boldsymbol{\lambda} = (\theta_0,\theta_1,\phi_{01},\phi_{11},\gamma,c_1,\ldots,c_m)'$.

3. The Exponential STAR(1)-GARCH(1,1) (or ESTAR(1)-GARCH(1,1)) model:

$$f_0(y_{t-1};\boldsymbol{\lambda}) = \theta_0 + \theta_1 G(y_{t-1};\gamma,c), \text{ and} \tag{10}$$

$$f_1(y_{t-1};\boldsymbol{\lambda}) = \phi_0 + \phi_1 G(y_{t-1};\gamma,c), \tag{11}$$

where

$$G\left(y_{t-1};\gamma,c\right) = 1 - e^{-\gamma(y_{t-1}-c)^2}, \tag{12}$$

with $\boldsymbol{\lambda} = (\theta_0,\theta_1,\phi_{01},\phi_{11},\gamma,c)'$.

3 Probabilistic Properties

ASSUMPTION 1. *The sequence $\{\eta_t\}$ of $\mathsf{IID}(0,1)$ random variables is drawn from a continuous (with respect to Lebesgue measure on the real line), unimodal, positive everywhere density, and bounded in a neighborhood of 0.*

ASSUMPTION 2. *The parameters of the model satisfy the following conditions:*
(R.1a) $\gamma_i > 0$, $i = 1,\ldots,m$, and $c_1 < c_2 < \cdots < c_m$ in (4) and (5);
(R1.b) $\gamma > 0$ and $c_1 \leq c_2 \leq \cdots \leq c_m$ in (7) and (8);
(R.1c) $\gamma > 0$ in (10) and (11);
(R.2) $\omega > 0$, $\alpha > 0$, and $\beta > 0$.

Assumption 1 is standard. Note that we do not assume symmetry of the distribution, which is particularly useful when modelling financial time series. The restrictions (R.1a)–(R.1c) in Assumption 2 are important to guarantee that the model is globally identifiable. Restriction (R.2) is a sufficient condition for $h_t > 0$ with probability one.

Note that $\mathbf{z}_t = (y_t, h_t, \eta_t)'$ is a Markov chain with homogenous transition probability expressed as

$$\mathbf{z}_t = \mathbf{F}\left(\mathbf{z}_{t-1}\right) + \mathbf{e}_t, \tag{13}$$

where

$$\mathbf{F}\left(\mathbf{z}_{t-1}\right)=\begin{pmatrix}\lambda_{0,t-1}+\lambda_{1,t-1}y_{t-1}\\ \omega+\left(\beta+\alpha\eta_{t-1}^{2}\right)\\ 0\end{pmatrix}$$

and $\mathbf{e}_t = (\varepsilon_t, 0, \eta_t)'$.

The following theorems state the necessary and sufficient conditions for strict stationarity and geometric ergodicity of the STAR-GARCH models considered in this paper.

THEOREM 1 (STATIONARITY – MRLSTAR(1)-GARCH(1,1) MODEL). *Define* $\overline{\theta} = \sum_{i=0}^{m} \theta_i$ *and* $\overline{\phi} = \sum_{i=0}^{m} \phi_i$. *Under Assumption 1, and if (R.1a) in Assumption 2 holds, the process* $\{y_t, t \in \mathbb{Z}\}$ *defined by equations (1)–(3), (4), and (5) is strictly stationary and geometrically ergodic if and only if* $\mathsf{E}\left[\log\left(\beta+\alpha\eta_t^2\right)\right] < 0$*, and one of the following restrictions holds:*

$$\phi_0 < 1,\ \overline{\phi} < 1,\ \textit{and}\ \phi_0\overline{\phi} < 1; \tag{14}$$

$$\phi_0 = 1,\ \overline{\phi} < 1,\ \textit{and}\ \theta_0 > 0; \tag{15}$$

$$\phi_0 < 1,\ \overline{\phi} = 1,\ \textit{and}\ \overline{\theta} < 0; \tag{16}$$

$$\phi_0 = 1,\ \overline{\phi} = 1,\ \textit{and}\ \overline{\theta} < 0 < \theta_0; \tag{17}$$

$$\phi_0\overline{\phi} = 1,\ \theta_1 < 0,\ \textit{and}\ \overline{\theta} + \overline{\phi}\theta_0 > 0. \tag{18}$$

Furthermore, the process $\{\mathbf{z}_t, t \in \mathbb{Z}\}$ *admits a unique causal expansion.*

THEOREM 2 (STATIONARITY – GSTAR(1)-GARCH(1,1) MODEL). *Set* $\overline{\theta} = \theta_0 + \theta_1$ *and* $\overline{\phi} = \phi_0 + \phi_1$. *Under Assumption 1, and if (R.1b) in Assumption 2 holds, the process* $\{y_t, t \in \mathbb{Z}\}$ *defined by equations (1)–(3), (7) and (8) is strictly stationary and geometrically ergodic if and only if* $\mathsf{E}\left[\log\left(\beta+\alpha\eta_t^2\right)\right] < 0$*, and one the following holds:*

1. m *is even and* $|\overline{\phi}| < 1$*, or*

2. m *is odd and one of the following conditions holds:*

$$\phi_0 < 1,\ \overline{\phi} < 1,\ \textit{and}\ \phi_0\overline{\phi} < 1; \tag{19}$$

$$\phi_0 = 1,\ \overline{\phi} < 1,\ \textit{and}\ \theta_0 > 0; \tag{20}$$

$$\phi_0 < 1,\ \overline{\phi} = 1,\ \textit{and}\ \overline{\theta} < 0; \tag{21}$$

$$\phi_0 = 1,\ \overline{\phi} = 1,\ \textit{and}\ \overline{\theta} < 0 < \theta_0; \tag{22}$$

$$\phi_0\overline{\phi} = 1,\ \theta_1 < 0,\ \textit{and}\ \overline{\theta} + \overline{\phi}\theta_0 > 0. \tag{23}$$

Furthermore, the process $\{\mathbf{z}_t, t \in \mathbb{Z}\}$ *admits a unique causal expansion.*

THEOREM 3 (STATIONARITY – ESTAR(1)-GARCH(1,1) MODEL). *Set* $\overline{\phi} = \phi_0 + \phi_1$. *Under Assumption 1, and if (R.1c) in Assumption 2 holds, the process* $\{y_t, t \in \mathbb{Z}\}$ *defined by equations (1)–*

(3), (10) and (11) is strictly stationary and geometrically ergodic if and only if $\mathsf{E}\left[\log\left(\beta+\alpha\eta_t^2\right)\right]<0$ *and* $\left|\overline{\phi}\right|<1$. *Furthermore, the process* $\{\mathbf{z}_t, t\in\mathbb{Z}\}$ *admits a unique causal expansion.*

If the conditions of the above theorems are met, the processes $\{y_t\}$ and $\{h_t\}$ have the following causal expansions:

$$y_t = \lambda_{0,t-1} + \sum_{j=1}^{\infty}\prod_{k=0}^{j-1}\left(\lambda_{0,t-1-j}\lambda_{1,t-1-k} + \lambda_{1,t-1-k}\varepsilon_{t-j}\right), \tag{24}$$

$$h_t = \omega\left[1 + \sum_{j=1}^{\infty}\prod_{k=1}^{j}\left(\beta + \alpha\eta_{t-i}^2\right)\right]. \tag{25}$$

THEOREM 4 (EXISTENCE OF MOMENTS – MRLSTAR(1)-GARCH(1,1) MODEL). *Define* $\overline{\phi}$ *as in Theorem 1. Assume* $\mathsf{E}\left[|\eta_t|^n\right]<\infty$. *If* $\theta_1<1$, $\overline{\phi}<1$, $\theta_1\overline{\phi}<1$, *and* $\mathsf{E}\left[\left(\beta+\alpha\eta_t^2\right)^n\right]<1$, *the invariant probability distribution for the MLSTAR(1)-GARCH(1,1) process has a finite* n*th-order moment.*

THEOREM 5 (EXISTENCE OF MOMENTS – GSTAR(1)-GARCH(1,1) MODEL). *Assume* $\mathsf{E}\left[|\eta_t|^k\right]<\infty$. *If* m *is even and* $|\overline{\phi}|<1$, *or* m *is odd and* $\phi_0<1$, $\overline{\phi}<1$, $\phi_0\overline{\phi}<1$, *and* $\mathsf{E}\left[\left(\beta+\alpha\eta_t^2\right)^n\right]<1$, *the invariant probability distribution for the GSTAR(1)-GARCH(1,1) process has a finite* n*th-order moment.*

THEOREM 6 (EXISTENCE OF MOMENTS – ESTAR(1)-GARCH(1,1) MODEL). *Assume* $\mathsf{E}\left[|\eta_t|^k\right]<\infty$. *If* $|\overline{\phi}|<1$ *and* $\mathsf{E}\left[\left(\beta+\alpha\eta_t^2\right)^n\right]<1$, *the invariant probability distribution for the ESTAR(1)-GARCH(1,1) process has a finite* n*th-order moment.*

4 Parameter Estimation and Asymptotic Theory

In this section we discuss the estimation of the STAR-GARCH models. Set $\boldsymbol{\psi}=\left(\boldsymbol{\lambda}',\boldsymbol{\pi}'\right)'$, where $\boldsymbol{\lambda}$ is the vector of parameters of the conditional mean, as defined in Section 2, and $\boldsymbol{\pi}=(\omega,\alpha,\beta)'$ is the vector of parameters of the conditional variance. As the distribution of η_t is unknown, the parameter vector $\boldsymbol{\psi}$ is estimated by the quasi-maximum likelihood (QML) method.

Consider the following assumption.

ASSUMPTION 3. *The true parameter vector* $\boldsymbol{\psi}_0\in\boldsymbol{\Psi}\subseteq\mathbb{R}^N$ *is in the interior of* $\boldsymbol{\Psi}$, *a compact and convex parameter space, where* $N=\mathsf{dim}(\boldsymbol{\lambda})+\mathsf{dim}(\boldsymbol{\pi})$ *is the total number of parameters.*

The quasi-log-likelihood function of the STAR-GARCH model is given by:

$$\begin{aligned}\mathcal{L}_T(\boldsymbol{\psi}) &= \frac{1}{T}\sum_{t=1}^{T} l_t(\boldsymbol{\psi}),\\ &= \frac{1}{T}\sum_{t=1}^{T} -\frac{1}{2}\ln(2\pi) - \frac{1}{2}\ln(h_t) - \frac{\varepsilon_t^2}{2h_t}.\end{aligned} \tag{26}$$

Note that the processes y_t and h_t, $t \leq 0$, are unobserved, and hence are only arbitrary constants. Thus, $\mathcal{L}_T(\boldsymbol{\psi})$ is a quasi-log-likelihood function that is not conditional on the true (y_0, h_0), making it suitable for practical applications.

However, to prove the asymptotic properties of the QMLE, it is more convenient to work with the unobserved process $\{(\varepsilon_{u,t}, h_{u,t}) : \ t = 0, \pm 1, \pm 2, \ldots\}$.

The unobserved quasi-log-likelihood function conditional on $\mathcal{F}_0 = (y_0, y_{-1}, y_{-2}, \ldots)$ is

$$\begin{aligned}\mathcal{L}_{u,T}(\boldsymbol{\psi}) &= \frac{1}{T}\sum_{t=1}^{T} l_{u,t}(\boldsymbol{\psi}), \\ &= \frac{1}{T}\sum_{t=1}^{T} -\frac{1}{2}\ln(2\pi) - \frac{1}{2}\ln(h_{u,t}) - \frac{\varepsilon_{u,t}^2}{2h_{u,t}}.\end{aligned} \tag{27}$$

The main difference between $\mathcal{L}_T(\boldsymbol{\psi})$ and $\mathcal{L}_{u,T}(\boldsymbol{\psi})$ is that the former is conditional on any initial values, whereas the latter is conditional on an infinite series of past observations. In practical situations, the use of (27) is not possible.

Let

$$\widehat{\boldsymbol{\psi}}_T = \underset{\boldsymbol{\psi}\in\boldsymbol{\Psi}}{\mathsf{argmax}}\mathcal{L}_T(\boldsymbol{\psi}) = \underset{\boldsymbol{\psi}\in\boldsymbol{\Psi}}{\mathsf{argmax}}\left(\frac{1}{T}\sum_{t=1}^{T} l_t(\boldsymbol{\psi})\right),$$

and

$$\widehat{\boldsymbol{\psi}}_{u,T} = \underset{\boldsymbol{\psi}\in\boldsymbol{\Psi}}{\mathsf{argmax}}\mathcal{L}_{u,T}(\boldsymbol{\psi}) = \underset{\boldsymbol{\psi}\in\boldsymbol{\Psi}}{\mathsf{argmax}}\left(\frac{1}{T}\sum_{t=1}^{T} l_{u,t}(\boldsymbol{\psi})\right).$$

Define $\mathcal{L}(\boldsymbol{\psi}) = \mathsf{E}\left[l_{u,t}(\boldsymbol{\psi})\right]$. In the following subsection, we discuss the existence of $\mathcal{L}(\boldsymbol{\psi})$ and the identifiability of the STAR-GARCH models. Then, in Subsection 4.2, we prove the consistency of $\widehat{\boldsymbol{\psi}}_T$ and $\widehat{\boldsymbol{\psi}}_{u,T}$. We first prove the strong consistency of $\widehat{\boldsymbol{\psi}}_{u,T}$, and then show that

$$\sup_{\boldsymbol{\psi}\in\boldsymbol{\Psi}} |\mathcal{L}_{u,T}(\boldsymbol{\psi}) - \mathcal{L}_T(\boldsymbol{\psi})| \stackrel{a.s.}{\to} 0,$$

so that the consistency of $\widehat{\boldsymbol{\psi}}_T$ follows. Asymptotic normality of both estimators is considered in Subsection 4.3. We prove the asymptotic normality of $\widehat{\boldsymbol{\psi}}_{u,T}$. The proof of $\widehat{\boldsymbol{\psi}}_T$ is straightforward.

4.1 Existence of the QMLE

The following theorem proves the existence of $\mathcal{L}(\boldsymbol{\psi})$. It is based on Theorem 2.12 in White (1994), which establishes that $\mathcal{L}(\boldsymbol{\psi})$ exists under certain conditions of continuity and measurability of the quasi-log-likelihood function.

THEOREM 7. *Under Assumptions 1 and 2, $\mathcal{L}(\boldsymbol{\psi})$ exists, is finite, and is uniquely maximized at $\boldsymbol{\psi}_0$ if:*

1. *$\{y_t, t \in \mathbb{Z}\}$ follows the process defined by (1)–(6) and the conditions of Theorem 1 hold;*

2. $\{y_t, t \in \mathbb{Z}\}$ *follows the process defined by (1)–(3), (7)–(9), and the conditions of Theorem 2 hold; or*

3. $\{y_t, t \in \mathbb{Z}\}$ *follows the process defined by (1)–(3), (10)–(12), and the conditions of Theorem 3 hold.*

4.2 Consistency

The following theorem states the sufficient conditions for strong consistency of the QMLE.

THEOREM 8. *Under Assumptions 1–3, the QMLE of ψ is strongly consistent for ψ_0, $\widehat{\psi} \xrightarrow{a.s.} \psi_0$, if*

1. $\{y_t, t \in \mathbb{Z}\}$ *follows the process defined by (1)–(6) and the conditions of Theorem 1 hold;*

2. $\{y_t, t \in \mathbb{Z}\}$ *follows the process defined by (1)–(3), (7)–(9), and the conditions of Theorem 2 hold; or*

3. $\{y_t, t \in \mathbb{Z}\}$ *follows the process defined by (1)–(3), (10)–(12), and the conditions of Theorem 3 hold.*

4.3 Asymptotic Normality

First, we introduce the following matrices:

$$\mathbf{A}(\psi_0) = \mathsf{E}\left[-\left.\frac{\partial^2 l_{u,t}(\psi)}{\partial\psi\partial\psi'}\right|_{\psi_0} \right], \quad \mathbf{B}(\psi_0) = \mathsf{E}\left[\left.\frac{\partial l_{u,t}(\psi)}{\partial\psi}\right|_{\psi_0} \left.\frac{\partial l_{u,t}(\psi)}{\partial\psi'}\right|_{\psi_0} \right],$$

and

$$\begin{aligned}\mathbf{A}_T(\psi) =& \frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{2h_t}\left(\frac{\varepsilon_t^2}{h_t}-1\right)\frac{\partial^2 h_t}{\partial\psi\partial\psi'} - \frac{1}{2h_t^2}\left(2\frac{\varepsilon_t^2}{h_t}-1\right)\frac{\partial h_t}{\partial\psi}\frac{\partial h_t}{\partial\psi'} \right. \\ & \left. + \left(\frac{\varepsilon_t}{h_t^2}\right)\left(\frac{\partial\varepsilon_t}{\partial\psi}\frac{\partial h_t}{\partial\psi'} + \frac{\partial h_t}{\partial\psi}\frac{\partial\varepsilon_t}{\partial\psi'}\right) + \frac{1}{h_t}\left(\frac{\partial\varepsilon_t}{\partial\psi}\frac{\partial\varepsilon_t}{\partial\psi'} + \varepsilon_t\frac{\partial^2\varepsilon_t}{\partial\psi}\right)\right]\end{aligned} \tag{28}$$

$$\begin{aligned}\mathbf{B}_T(\psi) &= \frac{1}{T}\sum_{t=1}^{T}\frac{\partial l_t(\psi)}{\partial\psi}\frac{\partial l_t(\psi)}{\partial\psi'} \\ &= \frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{4h_t^2}\left(\frac{\varepsilon_t^2}{h_t}-1\right)^2\frac{\partial h_t}{\partial\psi}\frac{\partial h_t}{\partial\psi'} + \frac{\varepsilon_t^2}{h_t}\frac{\partial\varepsilon_t}{\partial\psi}\frac{\partial\varepsilon_t}{\partial\psi'} \right. \\ &\qquad \left. - \frac{\varepsilon_t}{2h_t^2}\left(\frac{\varepsilon_t^2}{h_t}-1\right)\left(\frac{\partial h_t}{\partial\psi}\frac{\partial\varepsilon_t}{\partial\psi'} + \frac{\partial\varepsilon_t}{\partial\psi}\frac{\partial h_t}{\partial\psi'}\right)\right]\end{aligned} \tag{29}$$

Consider the additional assumption:

ASSUMPTION 4. *There exists no set* Λ *of cardinal 2 such that* $\mathsf{Pr}[\eta_t \in \Lambda] = 1$.

As in Francq and Zakoïan (2004), Assumption 4 is necessary for identifying reasons when the distribution of η_t is non-symmetric.

The following theorem states the asymptotic normality result.

THEOREM 9. *Under Assumptions 1–3, 4, the additional assumption* $\mathsf{E}\left[\varepsilon_t^4\right] = \mu_4 < \infty$, *and if either:*

1. $\{y_t, t \in \mathbb{Z}\}$ *follows the process defined by (1)–(6) and the conditions of Theorem 1 hold;*

2. $\{y_t, t \in \mathbb{Z}\}$ *follows the process defined by (1)–(3), (7)–(9), and the conditions of Theorem 2 hold; or*

3. $\{y_t, t \in \mathbb{Z}\}$ *follows the process defined by (1)–(3), (10)–(12), and the conditions of Theorem 3 hold.*

then

$$T^{1/2}(\widehat{\boldsymbol{\psi}}_T - \boldsymbol{\psi}_0) \xrightarrow{d} \mathsf{N}\left(\mathbf{0}, \boldsymbol{\Omega}\right), \tag{30}$$

where $\boldsymbol{\Omega} = \mathbf{A}(\boldsymbol{\psi}_0)^{-1}\mathbf{B}(\boldsymbol{\psi}_0)\mathbf{A}(\boldsymbol{\psi}_0)^{-1}$. *If the distribution of* η_t *is symmetric and* $\mathsf{E}\left[\eta_t^4\right] = \kappa_4$, *then*

$$\mathbf{A}(\boldsymbol{\psi}_0) = \begin{pmatrix} \mathbf{A}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2 \end{pmatrix}, \; \mathbf{B}(\boldsymbol{\psi}_0) = \begin{pmatrix} \mathbf{B}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_2 \end{pmatrix}, \text{ with}$$

$$\mathbf{A}_1 = \mathsf{E}\left[\left.\frac{1}{h_t^2}\frac{\partial h_t}{\partial \boldsymbol{\lambda}}\frac{\partial h_t}{\partial \boldsymbol{\lambda}'}\right|_{\boldsymbol{\psi}_0}\right] + \mathsf{E}\left[\left.\frac{2}{h_t^2}\frac{\partial \varepsilon_t}{\partial \boldsymbol{\lambda}}\frac{\partial \varepsilon_t}{\partial \boldsymbol{\lambda}'}\right|_{\boldsymbol{\psi}_0}\right],$$

$$\mathbf{A}_2 = \mathsf{E}\left[\left.\frac{1}{h_t^2}\frac{\partial h_t}{\partial \boldsymbol{\pi}}\frac{\partial h_t}{\partial \boldsymbol{\pi}'}\right|_{\boldsymbol{\psi}_0}\right],$$

$$\mathbf{B}_1 = (\kappa_4 - 1)\mathsf{E}\left[\left.\frac{1}{h_t^2}\frac{\partial h_t}{\partial \boldsymbol{\lambda}}\frac{\partial h_t}{\partial \boldsymbol{\lambda}'}\right|_{\boldsymbol{\psi}_0}\right] + 4\mathsf{E}\left[\left.\frac{1}{h_t^2}\frac{\partial \varepsilon_t}{\partial \boldsymbol{\lambda}}\frac{\partial \varepsilon_t}{\partial \boldsymbol{\lambda}'}\right|_{\boldsymbol{\psi}_0}\right], \text{ and}$$

$$\mathbf{B}_2 = (\kappa_4 - 1)\mathsf{E}\left[\left.\frac{1}{h_t^2}\frac{\partial h_t}{\partial \boldsymbol{\pi}}\frac{\partial h_t}{\partial \boldsymbol{\pi}'}\right|_{\boldsymbol{\psi}_0}\right].$$

Furthermore, the matrices $\mathbf{A}(\boldsymbol{\psi}_0)$ *and* $\mathbf{B}(\boldsymbol{\psi}_0)$ *are consistently estimated by* $\mathbf{A}_T(\widehat{\boldsymbol{\psi}})$ *and* $\mathbf{B}_T(\widehat{\boldsymbol{\psi}})$, *respectively.*

5 Concluding Remarks

In this paper we have derived the necessary and sufficient conditions for strict stationarity and geometric ergodicity of three different classes of first-order STAR-GARCH models, and the sufficient conditions for the existence of moments. This is important in order to find the conditions under which the traditional LM linearity tests are valid. The asymptotic properties of the QMLE have also been considered. We have proved that the QMLE is strongly consistent and asymptotically normal under weak conditions. These new results should be important for the estimation of STAR-GARCH models in financial econometrics.

Appendix

A Proofs of Theorems 1– 6

The proofs of the theorems are based on Chan, Petruccelli, Tong and Woolford (1985), and makes use of the results in Tweedie (1988).

Define $\mathbf{z}_t$ as in (13). If there is a compact set $\mathcal{A} \in \mathcal{B}$, where $\mathcal{B}$ is the Borel σ-algebra in $\mathbb{R}^3$, and a non-negative measurable function $g : \mathbb{R}^3 \rightarrow \mathbb{R}$, such that g is bounded away from zero and infinity in $\mathcal{A}$ and

$$\sup_{\mathbf{z}\in\mathcal{A}} \mathsf{E}\left[g(\mathbf{z}_t)|\mathbf{z}_{t-1} = \mathbf{z}\right] < \infty, \text{ and} \tag{A.1}$$

$$\mathsf{E}\left[g(\mathbf{z}_t)|\mathbf{z}_{t-1} = \mathbf{z}\right] < g(\mathbf{z}) - 1, \ \mathbf{z} \in \mathcal{A}^c, \tag{A.2}$$

where $\mathcal{A}^c$ is the complement of $\mathcal{A}$, then the process $\mathbf{z}_t$ is strictly stationary and geometrically ergodic.

A.1 Proof of Theorem 1

For purposes of the present paper, the most important features of the transition function are that it is a monotonically increasing function with an inflection point at c_i, $i = 1, \ldots, m$, and that it is limited by 0 when $y_{t-1} \rightarrow -\infty$ and by 1 when $y_{t-1} \rightarrow \infty$, since $\gamma_i > 0$, $i = 1, \ldots, m$ (Assumption 2).

First, consider a compact set $\mathcal{A} = [-M, M] \times [0, M] \times [-M, M]$, where M is a positive constant such that $M >> c_m$ and $-M << c_1$. Hence, given an arbitrarily small positive number δ, $G(y; \gamma_1, c_1) \leq \delta$ if $y_{t-1} < -M$ and $|G(y; \gamma_1, c_1) - 1| \leq \delta$ if $y_{t-1} > M$.

Set $g(\mathbf{z}_t) = v(y_t) + h_t + |\eta_t|$. Following Nelson (1990), and under the assumptions that $\mathsf{E}[|\eta_t|] < \infty$ and $\mathsf{E}\left[|\eta_t|^2\right] < \infty$, it is clear that $\mathsf{E}[h_t] < \infty$ and $\mathsf{E}[|\varepsilon_t|] < \infty$ if and only if $\mathsf{E}\left[\log\left(\beta + \alpha\eta_t^2\right)\right] < 0$. Thus, in order to show that $\mathbf{z}_t$ is ergodic for each of the conditions (14)–(18) in Theorem 1, it is sufficient to show that

$$\sup_{y\in[-M,M]} \mathsf{E}\left[v(y_t)|y_{t-1} = y\right] < \infty, \text{ and} \tag{A.3}$$

$$\mathsf{E}\left[v(y_t)|y_{t-1} = y\right] < g(y) - 1, \ y \notin [-M, M]. \tag{A.4}$$

As $G(y; \gamma_i, c_i) \rightarrow 0$ as $y \rightarrow -\infty$ and $G(y; \gamma_i, c_i) \rightarrow 1$ as $y \rightarrow \infty$, $i = 1, \ldots, M$, the result follows by using Theorem 2.1 in Chan, Petruccelli, Tong and Woolford (1985). ■

A.2 Proof of Theorem 2

It is clear that, when m is odd, the transition function is limited by 0 when $y_{t-1} \rightarrow -\infty$ and by 1 when $y_{t-1} \rightarrow \infty$, since $\gamma > 0$ (Assumption 2). The proof follows the same reasoning as in the proof of Theorem 1.

When m is even, the transition function is limited by 1 when $y_{t-1} \to \pm\infty$. Again, the proof follows along the same lines as the proof of Theorem 1. ■

A.3 Proof of Theorem 3

It is clear that the transition function is limited by 1 when $y_{t-1} \to \pm\infty$. Again, the proof follows the same reasoning as in the proof of Theorem 1. ■

A.4 Proof of Theorem 4

From Ling and McAleer (2002b), $\mathsf{E}\left[\left(\beta + \alpha\eta_t^2\right)^n\right] < 1$ is the necessary and sufficient condition for $\mathsf{E}\left[\varepsilon_t^n\right] < \infty$. The remainder of the proof is similar to the proof of Theorem 2.3 in Chan, Petruccelli, Tong and Woolford (1985). ■

A.5 Proof of Theorem 5

The proof is similar to the proof of Theorem 4. ■

A.6 Proof of Theorem 6

The proof is similar to the proof of Theorem 4. ■

B Proofs of Theorems 7–9

B.1 Proof of Theorem 7

It is easy to see that $\mathbf{F}(\mathbf{z}_t)$, as in (13), is a continuous function in the parameter vector $\boldsymbol{\psi}$. Similarly, we can see that $\mathbf{F}(\mathbf{z}_t)$ is continuous in $\mathbf{z}_t$, and therefore is measurable, for each fixed value of $\boldsymbol{\psi}$.

Furthermore, under the restrictions in Assumption 2, and if the stationarity conditions of either Theorem 1, 2, or 3 are satisfied, then $\mathsf{E}\left[\sup_{\boldsymbol{\psi}\in\boldsymbol{\Psi}} |h_{u,t}|\right] < \infty$ and $\mathsf{E}\left[\sup_{\boldsymbol{\psi}\in\boldsymbol{\Psi}} |y_{u,t}|\right] < \infty$. By Jensen´s inequality, $\mathsf{E}\left[\sup_{\boldsymbol{\psi}\in\boldsymbol{\Psi}} |\ln|h_{u,t}||\right] < \infty$. Thus, $\mathsf{E}\left[|l_{u,t}(\boldsymbol{\psi})|\right] < \infty \; \forall\, \boldsymbol{\psi} \in \boldsymbol{\Psi}$.

Let $h_{0,t}$ be the true conditional variance and $\varepsilon_{0,t} = h_{0,t}^{1/2}\eta_t$. In order to show that $\mathcal{L}(\boldsymbol{\psi})$ is uniquely maximized at $\boldsymbol{\psi}_0$, rewrite the maximization problem as

$$\max_{\boldsymbol{\psi}\in\boldsymbol{\Psi}} \left[\mathcal{L}(\boldsymbol{\psi}) - \mathcal{L}(\boldsymbol{\psi}_0)\right] = \max_{\boldsymbol{\psi}\in\boldsymbol{\Psi}} \left\{\mathsf{E}\left[\ln\left(\frac{h_{0,t}}{h_{u,t}}\right) - \frac{\varepsilon_t^2}{h_{u,t}} + 1\right]\right\}. \tag{B.5}$$

Writing $\varepsilon_t = \varepsilon_t - \varepsilon_{0,t} + \varepsilon_{0,t}$, equation (B.5) becomes

$$\begin{aligned}\max_{\boldsymbol{\psi}\in\boldsymbol{\Psi}} \left[\mathcal{L}(\boldsymbol{\psi}) - \mathcal{L}(\boldsymbol{\psi}_0)\right] &= \max_{\boldsymbol{\psi}\in\boldsymbol{\Psi}} \Bigg\{ \mathsf{E}\left[\ln\left(\frac{h_{0,t}}{h_{u,t}}\right) - \frac{h_{0,t}}{h_{u,t}} + 1\right] - \mathsf{E}\left[\frac{\left[\varepsilon_t - \varepsilon_{0,t}\right]^2}{h_{u,t}}\right] \\ &\qquad - \mathsf{E}\left[\frac{2\eta_t h_{0,t}^{1/2}\left(\varepsilon_t - \varepsilon_{0,t}\right)}{h_{u,t}}\right] \Bigg\} \\ &= \max_{\boldsymbol{\psi}\in\boldsymbol{\Psi}} \left\{ \mathsf{E}\left[\ln\left(\frac{h_{0,t}}{h_{u,t}}\right) - \frac{h_{0,t}}{h_{u,t}} + 1\right] - \mathsf{E}\left[\frac{\left[\varepsilon_t - \varepsilon_{0,t}\right]^2}{h_{u,t}}\right] \right\},\end{aligned} \tag{B.6}$$

where

$$\mathsf{E}\left[\frac{2\eta_t h_{0,t}^{1/2}\left(\varepsilon_t - \varepsilon_{0,t}\right)}{h_{u,t}}\right] = 0$$

by the Law of Iterated Expectations.

Note that, for any $x > 0$, $m(x) = \ln(x) - x \leq 0$, so that

$$\mathsf{E}\left[\ln\left(\frac{h_{0,t}}{h_{u,t}}\right) - \frac{h_{0,t}}{h_{u,t}}\right] \leq 0.$$

Furthermore, $m(x)$ is maximized at $x = 1$. If $x \neq 1$, $m(x) < m(1)$, implying that $\mathsf{E}[m(x)] \leq \mathsf{E}[m(1)]$, with equality only if $x = 1$ a.s.. However, this will occur only if $\frac{h_{0,t}}{h_{u,t}} = 1$, a.s.. In addition,

$$\mathsf{E}\left[\frac{\left[\varepsilon_t - \varepsilon_{0,t}\right]^2}{h_{u,t}}\right] = 0$$

if and only if $\varepsilon_t = \varepsilon_{0,t}$. Hence, $\boldsymbol{\psi} = \boldsymbol{\psi}_0$. This completes the proof. ■

B.2 Proof of Theorem 8

Following White (1994), Theorem 3.5, $\widehat{\boldsymbol{\psi}}_{u,T} \overset{a.s.}{\to} \boldsymbol{\psi}_0$ if the following conditions hold:

(1) The parameter space $\boldsymbol{\Psi}$ is compact.

(2) $\mathcal{L}_{u,T}(\boldsymbol{\psi})$ is continuous in $\boldsymbol{\psi} \in \boldsymbol{\Psi}$. Furthermore, $\mathcal{L}_{u,T}(\boldsymbol{\psi})$ is a measurable function of y_t, $t = 1, \ldots, T$, for all $\boldsymbol{\psi} \in \boldsymbol{\Psi}$.

(3) $\mathcal{L}(\boldsymbol{\psi})$ has a unique maximum at $\boldsymbol{\psi}_0$.

(4) $\lim_{T\to\infty} \sup_{\boldsymbol{\psi}\in\boldsymbol{\Psi}} |\mathcal{L}_{u,T}(\boldsymbol{\psi}) - \mathcal{L}(\boldsymbol{\psi})| = 0,\ a.s.$.

Condition (1) holds by assumption. Theorem 7 shows that Conditions (2) and (3) are satisfied. By Lemma 1, Condition (4) is also satisfied. Thus, $\widehat{\boldsymbol{\psi}}_{u,T} \overset{a.s.}{\to} \boldsymbol{\psi}_0$.

Lemma 2 shows that

$$\lim_{T\to\infty} \sup_{\boldsymbol{\psi}\in\boldsymbol{\Psi}} |\mathcal{L}_{u,T}(\boldsymbol{\psi}) - \mathcal{L}_T(\boldsymbol{\psi})| = 0\, a.s.,$$

implying that $\widehat{\boldsymbol{\psi}}_T \overset{a.s.}{\to} \boldsymbol{\psi}_0$. This completes the proof. ■

B.3 Proof of Theorem 9

We start by proving asymptotic normality of the QMLE using the unobserved log-likelihood. When this is shown, the proof using the observed log-likelihood is immediate by Lemmas 2 and 4. According to Theorem 6.4 in White (1994), to prove the asymptotic normality of the QMLE we need the following conditions in addition to those stated in the proof of Theorem 8:

(5) The true parameter vector $\boldsymbol{\psi}_0$ is interior to $\boldsymbol{\Psi}$.

(6) The matrix

$$\mathbf{A}_T(\boldsymbol{\psi}) = \frac{1}{T}\sum_{t=1}^{T}\left(\frac{\partial^2 l_t(\boldsymbol{\psi})}{\partial\boldsymbol{\psi}\partial\boldsymbol{\psi}'}\right)$$

exists $a.s.$ and is continuous in $\boldsymbol{\Psi}$.

(7) The matrix $\mathbf{A}_T(\boldsymbol{\psi}) \overset{a.s.}{\to} \mathbf{A}(\boldsymbol{\psi}_0)$, for any sequence $\boldsymbol{\psi}_T$, such that $\boldsymbol{\psi}_T \overset{a.s.}{\to} \boldsymbol{\psi}_0$.

(8) The score vector satisfies

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(\frac{\partial l_t(\boldsymbol{\psi})}{\partial\boldsymbol{\psi}}\right) \overset{d}{\to} \mathsf{N}(\mathbf{0}, \mathbf{B}(\boldsymbol{\psi}_0)).$$

Condition (5) is satisfied by assumption. Condition (6) follows from the fact that $l_t(\boldsymbol{\psi})$ is differentiable of order two on $\boldsymbol{\psi} \in \boldsymbol{\Psi}$, and the stationarity of the STAR-GARCH model. The non-singularity of $\mathbf{A}(\boldsymbol{\psi}_0)$ and $\mathbf{B}(\boldsymbol{\psi}_0)$ follows from Lemma 4. Furthermore, Lemmas 3 and 5 implies that Condition (7) is satisfied. In Lemma 6 below, we prove that condition (8) is also satisfied. This completes the proof. ■

C Lemmas

LEMMA 1. *Suppose that y_t follows a STAR-GARCH model satisfying the restrictions in Assumptions 1 and 2, and the stationarity and ergodicity conditions are met. Then,*

$$\lim_{T\to\infty}\sup_{\boldsymbol{\psi}\in\boldsymbol{\Psi}} |\mathcal{L}_{u,T}(\boldsymbol{\psi}) - \mathcal{L}(\boldsymbol{\psi})| = 0,\ a.s..$$

PROOF. Set $g(\mathbf{Y}_t, \boldsymbol{\psi}) = l_{u,t}(\boldsymbol{\psi}) - \mathsf{E}\left[l_{u,t}(\boldsymbol{\psi})\right]$, where $\mathbf{Y}_t = [y_t, y_{t-1}, y_{t-2}, \ldots]'$. Hence, $\mathsf{E}\left[g(\mathbf{Y}_t, \boldsymbol{\psi})\right] = 0$. It is clear that $\mathsf{E}\left[\sup_{\boldsymbol{\psi}\in\boldsymbol{\Psi}} |g(\mathbf{Y}_t, \boldsymbol{\psi})|\right] < \infty$ by Theorem 7. Furthermore, as $g(\mathbf{Y}_t, \boldsymbol{\psi})$ is strictly stationary and ergodic, then $\lim_{T\to\infty}\sup_{\boldsymbol{\psi}\in\boldsymbol{\Psi}} \left|T^{-1}\sum_{t=1}^{T} g(\mathbf{Y}_t, \boldsymbol{\psi})\right| = 0,\ a.s.$. This completes the proof. ■

LEMMA 2. *Under the assumptions of Lemma 1,*

$$\lim_{T\to\infty}\sup_{\boldsymbol{\psi}\in\boldsymbol{\Psi}} |\mathcal{L}_{u,T}(\boldsymbol{\psi}) - \mathcal{L}_T(\boldsymbol{\psi})| = 0, a.s..$$

PROOF. First, write

$$h_t = \sum_{i=0}^{t-1}\beta^i\left(\omega + \alpha\varepsilon_{t-1-i}^2\right) + \beta^t h_0 \text{ and}$$

$$h_{u,t} = \beta^{t-1}\left(\omega + \alpha\varepsilon_{u,0}^2\right) + \sum_{i=0}^{t-2}\beta^i\left(\omega + \alpha\varepsilon_{t-1-i}^2\right) + \beta^t h_{u,0},$$

such that

$$|h_t - h_{u,t}| = |\beta^{t-1}\alpha\left(\varepsilon_0^2 - \varepsilon_{u,0}^2\right) + \beta^t\left(h_0 - h_{u,0}\right)|$$
$$\leq \beta^{t-1}\alpha\left|\varepsilon_0^2 - \varepsilon_{u,0}^2\right| + \beta^t\left|h_0 - h_{u,0}\right|.$$

Under the stationarity of the process, and if (R.2) in Assumption 2 and the log-moment condition hold, it is clear that $0 < \beta < 1$. Furthermore, $h_{u,0}$ and $\varepsilon_{0,u}^2$ are well defined, as

$$\mathsf{Pr}\left[\sup_{\boldsymbol{\psi}\in\boldsymbol{\Psi}}\left(h_{u,0} > K_1\right)\right] \to 0 \text{ as } K_1 \to \infty, \text{ and } \mathsf{Pr}\left[\sup_{\boldsymbol{\psi}\in\boldsymbol{\Psi}}\left(\varepsilon_{u,0}^2 > K_2\right)\right] \to 0 \text{ as } K_2 \to \infty.$$

Thus,

$$\sup_{\boldsymbol{\psi}\in\boldsymbol{\Psi}}|h_t - h_{u,t}| \leq K_h\rho_1^t,\ a.s., \text{and}$$

$$\sup_{\boldsymbol{\psi}\in\boldsymbol{\Psi}}\left|\varepsilon_0^2 - \varepsilon_{u,0}^2\right| \leq K_\varepsilon\rho_2^t,\ a.s.,$$

where K_h and K_ε are positive and finite constants, $0 < \rho_1 < 1$, and $0 < \rho_2 < 1$. Hence, as $h_t > \omega$ and $\log(x) \leq x - 1$,

$$\sup_{\boldsymbol{\psi}\in\boldsymbol{\Psi}}|l_t - l_{u,t}| \leq \sup_{\boldsymbol{\psi}\in\boldsymbol{\Psi}}\left[\varepsilon_t^2\left|\frac{h_{u,t} - h_t}{h_t h_{u,t}}\right| + \left|\log\left(1 + \frac{h_t - h_{u,t}}{h_{u,t}}\right)\right|\right]$$
$$\leq \sup_{\boldsymbol{\psi}\in\boldsymbol{\Psi}}\left(\frac{1}{\omega^2}\right)K_h\rho_1^t\varepsilon_t^2 + \sup_{\boldsymbol{\psi}\in\boldsymbol{\Psi}}\left(\frac{1}{\omega}\right)K_h\rho_1^t,\ a.s..$$

Following the same arguments as in the proof of Theorems 2.1 and 3.1 in Francq and Zakoïan (2004), it can be shown that

$$\lim_{T\to\infty}\sup_{\boldsymbol{\psi}\in\boldsymbol{\Psi}}|\mathcal{L}_{u,T}(\boldsymbol{\psi}) - \mathcal{L}_T(\boldsymbol{\psi})| = 0, a.s..$$

This completes the proof. ■

LEMMA 3. *Under the conditions of Theorem 9,*

$$\mathsf{E}\left[\left|\left.\frac{\partial l_t(\boldsymbol{\psi})}{\partial\boldsymbol{\psi}}\right|_{\boldsymbol{\psi}_0}\right|\right] < \infty, \tag{C.7}$$

$$\mathsf{E}\left[\left|\left.\frac{\partial l_t(\boldsymbol{\psi})}{\partial\boldsymbol{\psi}}\right|_{\boldsymbol{\psi}_0}\left.\frac{\partial l_t(\boldsymbol{\psi})}{\partial\boldsymbol{\psi}'}\right|_{\boldsymbol{\psi}_0}\right|\right] < \infty,\ \textit{and} \tag{C.8}$$

$$\mathsf{E}\left[\left|\left.\frac{\partial^2 l_t(\boldsymbol{\psi})}{\partial\boldsymbol{\psi}\partial\boldsymbol{\psi}'}\right|_{\boldsymbol{\psi}_0}\right|\right] < \infty. \tag{C.9}$$

PROOF. Set

$$\nabla_0 l_{u,t} \equiv \left.\frac{\partial l_{u,t}(\boldsymbol{\psi})}{\partial\boldsymbol{\psi}}\right|_{\boldsymbol{\psi}_0},\ \ \nabla_0 h_{u,t} \equiv \left.\frac{\partial h_{u,t}}{\partial\boldsymbol{\psi}}\right|_{\boldsymbol{\psi}_0},\ \ \nabla_0\varepsilon_t \equiv \left.\frac{\partial\varepsilon_t}{\partial\boldsymbol{\psi}}\right|_{\boldsymbol{\psi}_0},$$

$$\nabla_0^2 l_{u,t} \equiv \left.\frac{\partial^2 l_{u,t}(\boldsymbol{\psi})}{\partial\boldsymbol{\psi}\partial\boldsymbol{\psi}'}\right|_{\boldsymbol{\psi}_0},\ \ \nabla_0^2 h_{u,t} \equiv \left.\frac{\partial^2 h_{u,t}}{\partial\boldsymbol{\psi}\partial\boldsymbol{\psi}'}\right|_{\boldsymbol{\psi}_0},\ \text{and}\ \nabla_0^2\varepsilon_t \equiv \left.\frac{\partial^2\varepsilon_t}{\partial\boldsymbol{\psi}\partial\boldsymbol{\psi}'}\right|_{\boldsymbol{\psi}_0}.$$

Then,

$$\nabla_0 l_{u,t} = \frac{1}{2h_{u,t}}\left(\frac{\varepsilon_t^2}{h_{u,t}} - 1\right)\nabla_0 h_{u,t} - \frac{\varepsilon_t}{h_{u,t}}\nabla_0\varepsilon_t$$

and

$$\begin{aligned}\nabla_0^2 l_{u,t} = &\left(\frac{\varepsilon_t^2}{h_{u,t}} - 1\right)\frac{1}{2h_{u,t}}\nabla_0^2 h_{u,t} - \frac{1}{2h_{u,t}^2}\left(2\frac{\varepsilon_t^2}{h_{u,t}} - 1\right)\nabla_0 h_{u,t}\nabla_0 h_{u,t}' \\ &+ \left(\frac{\varepsilon_t}{h_{u,t}^2}\right)\left(\nabla_0\varepsilon_t\nabla_0 h_{u,t}' + \nabla_0 h_{u,t}\nabla_0\varepsilon_t'\right) + \frac{1}{h_{u,t}}\left(\nabla_0\varepsilon_t\nabla_0\varepsilon_t' + \varepsilon_t\nabla_0^2\varepsilon_t\right).\end{aligned}$$

Set $\boldsymbol{\psi} = (\boldsymbol{\lambda}', \boldsymbol{\pi}')'$, where, as stated before, $\boldsymbol{\lambda}$ is the vector of parameters of the conditional mean and $\boldsymbol{\pi}$ is the vector of parameters of the conditional variance. As in the proof of Theorem 3.2 in Francq and Zakoïan (2004), the derivatives with respect to $\boldsymbol{\pi}$ are clearly bounded. We proceed by analyzing the derivatives with respect to $\boldsymbol{\lambda}$. As $\varepsilon_t = y_t - f_0(y_{t-1};\boldsymbol{\lambda}) - f_1(y_{t-1};\boldsymbol{\lambda})y_{t-1}$, we have

$$\frac{\partial\varepsilon_t}{\partial\boldsymbol{\lambda}} = -\frac{\partial f_0(y_{t-1};\boldsymbol{\lambda})}{\partial\boldsymbol{\lambda}} - \frac{\partial f_1(y_{t-1};\boldsymbol{\lambda})}{\partial\boldsymbol{\lambda}}y_{t-1}, \tag{C.10}$$

$$\frac{\partial^2\varepsilon_t}{\partial\boldsymbol{\lambda}\partial\boldsymbol{\lambda}'} = -\frac{\partial^2 f_0(y_{t-1};\boldsymbol{\lambda})}{\partial\boldsymbol{\lambda}\partial\boldsymbol{\lambda}'} - \frac{\partial^2 f_1(y_{t-1};\boldsymbol{\lambda})}{\partial\boldsymbol{\lambda}\partial\boldsymbol{\lambda}'}y_{t-1}, \tag{C.11}$$

$$\frac{\partial h_{u,t}}{\partial\boldsymbol{\lambda}} = 2\alpha\sum_{i=0}^{\infty}\left(\beta^i\varepsilon_{t-1-i}\frac{\partial\varepsilon_{t-1-i}}{\partial\boldsymbol{\lambda}}\right), \text{ and} \tag{C.12}$$

$$\frac{\partial^2 h_{u,t}}{\partial\boldsymbol{\lambda}\partial\boldsymbol{\lambda}'} = 2\alpha\sum_{i=0}^{\infty}\beta^i\left(\varepsilon_{t-1-i}\frac{\partial^2\varepsilon_{t-1-i}}{\partial\boldsymbol{\lambda}\partial\boldsymbol{\lambda}'} + \frac{\partial\varepsilon_{t-1-i}}{\partial\boldsymbol{\lambda}}\frac{\partial\varepsilon_{t-1-i}}{\partial\boldsymbol{\lambda}'}\right). \tag{C.13}$$

As the derivatives of the transition function are bounded, if the strict stationarity and ergodicity conditions hold, (C.10)–(C.13) are clearly bounded. Hence, the remainder of the proof follows from the proof of Theorem 3.2 (part (i)) in Francq and Zakoïan (2004). This completes the proof. ■

LEMMA 4. *Under the conditions of Theorem 9,* $\mathbf{A}(\boldsymbol{\psi}_0)$ *and* $\mathbf{B}(\boldsymbol{\psi}_0)$ *are nonsingular and, when* η_t *has a symmetric distribution, are block-diagonal.*

PROOF. First, note that (R1a)–(R1c) in Assumption 2 and Assumption 4 guarantee the minimality (identifiability) of the different specifications of the STAR models considered in this paper. Therefore, the results follow from the proof of Theorem 3.2 (part (ii)) in Francq and Zakoïan (2004). This completes the proof. ■

LEMMA 5. *Under the conditions of Theorem 9,*

(a) $$\lim_{T\to\infty}\sup_{\boldsymbol{\psi}\in\boldsymbol{\Psi}}\left\|\frac{1}{T}\sum_{t=1}^{T}\left[\frac{\partial l_{u,t}(\boldsymbol{\psi})}{\partial\boldsymbol{\psi}} - \frac{\partial l_t(\boldsymbol{\psi})}{\partial\boldsymbol{\psi}}\right]\right\| = \mathbf{0},\ a.s.,$$

(b) $$\lim_{T\to\infty}\sup_{\boldsymbol{\psi}\in\boldsymbol{\Psi}}\left\|\frac{1}{T}\sum_{t=1}^{T}\left[\frac{\partial^2 l_{u,t}(\boldsymbol{\psi})}{\partial\boldsymbol{\psi}\partial\boldsymbol{\psi}'} - \frac{\partial^2 l_t(\boldsymbol{\psi})}{\partial\boldsymbol{\psi}\partial\boldsymbol{\psi}'}\right]\right\| = \mathbf{0},\ a.s,\ \text{ and}$$

(c) $$\lim_{T\to\infty}\sup_{\boldsymbol{\psi}\in\boldsymbol{\Psi}}\left\|\frac{1}{T}\sum_{t=1}^{T}\frac{\partial^2 l_{u,t}(\boldsymbol{\psi})}{\partial\boldsymbol{\psi}\partial\boldsymbol{\psi}'} - \mathsf{E}\left[\frac{\partial^2 l_{u,t}(\boldsymbol{\psi})}{\partial\boldsymbol{\psi}\partial\boldsymbol{\psi}'}\right]\right\| = \mathbf{0},\ a.s..$$

PROOF.

First, assume that h_0 and $h_{u,0}$ are fixed constants. It is easy to show that

$$\begin{aligned}\left|\frac{\partial h_t}{\partial \boldsymbol{\lambda}} - \frac{\partial h_{u,t}}{\partial \boldsymbol{\lambda}}\right| &= 2\alpha\beta^{t-1}\left|\varepsilon_0\frac{\partial \varepsilon_0}{\partial \boldsymbol{\lambda}} - \varepsilon_{u,0}\frac{\partial \varepsilon_{u,0}}{\partial \boldsymbol{\lambda}}\right| \\ &\leq 2\alpha\beta^{t-1}\left(\left|\varepsilon_0\frac{\partial \varepsilon_0}{\partial \boldsymbol{\lambda}}\right| + \left|\varepsilon_{u,0}\frac{\partial \varepsilon_{u,0}}{\partial \boldsymbol{\lambda}}\right|\right) < \infty,\end{aligned}$$

as $0 < \beta < 1$ and y_t is stationary and ergodic. Hence, following the same arguments as in the proof of Theorem 3.2 (part (iii)) in Francq and Zakoïan (2004), it is straightforward to show that

$$\lim_{T\to\infty}\sup_{\boldsymbol{\psi}\in\boldsymbol{\Psi}}\left\|\frac{1}{T}\sum_{t=1}^{T}\left[\frac{\partial l_{u,t}(\boldsymbol{\psi})}{\partial \boldsymbol{\lambda}} - \frac{\partial l_t(\boldsymbol{\psi})}{\partial \boldsymbol{\lambda}}\right]\right\| = \mathbf{0}.$$

Furthermore, as

$$\begin{aligned}&\frac{\partial h_t}{\partial \omega} - \frac{\partial h_{u,t}}{\partial \omega} = 0 \\ &\frac{\partial h_t}{\partial \alpha} - \frac{\partial h_{u,t}}{\partial \alpha} = \varepsilon_0^2 - \varepsilon_{u,0}^2 \\ &\frac{\partial h_t}{\partial \beta} - \frac{\partial h_{u,t}}{\partial \beta} = (t-1)\beta^{t-2}\left(\varepsilon_0^2 - \varepsilon_{u,0}^2\right) + t\beta^{t-1}\left(h_0 - h_{u,0}\right),\end{aligned}$$

it is clear that

$$\lim_{T\to\infty}\sup_{\boldsymbol{\psi}\in\boldsymbol{\Psi}}\left\|\frac{1}{T}\sum_{t=1}^{T}\left[\frac{\partial l_{u,t}(\boldsymbol{\psi})}{\partial \boldsymbol{\pi}} - \frac{\partial l_t(\boldsymbol{\psi})}{\partial \boldsymbol{\pi}}\right]\right\| = \mathbf{0}.$$

The proof of part (a) is now complete. The proof of part (b) follows along similar lines. The proof of part (c) follows the same arguments as in the proof of Theorem 3.2 (part (v)) in Francq and Zakoïan (2004). This completes the proof. ■

LEMMA 6. *Under the conditions of Theorem 9,*

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left.\frac{\partial l_t(\boldsymbol{\psi})}{\partial \boldsymbol{\psi}}\right|_{\boldsymbol{\psi}_0} \xrightarrow{d} \mathsf{N}(\mathbf{0}, \mathbf{B}(\boldsymbol{\psi}_0)).$$

PROOF. Let $S_T = \sum_{t=1}^{T}\mathbf{c}'\nabla_0 l_{u,t}$, where $\mathbf{c}$ is a constant vector. Then S_T is a martingale with respect to $\mathcal{F}_t$, the filtration generated by all past observations of y_t. By the given assumptions, $\mathsf{E}\left[S_T\right] > 0$. Using the central limit theorem of Stout (1974),

$$T^{-1/2}S_T \xrightarrow{d} \mathsf{N}\left(0, \mathbf{c}'\mathbf{B}(\boldsymbol{\psi}_0)\mathbf{c}\right).$$

By the Cramér-Wold device,

$$T^{-1/2}\sum_{t=1}^{T}\left.\frac{\partial l_{u,t}(\boldsymbol{\psi})}{\partial \boldsymbol{\psi}}\right|_{\boldsymbol{\psi}_\mathbf{0}} \xrightarrow{d} \mathsf{N}\left(0, \mathbf{B}(\boldsymbol{\psi}_0)\right).$$

By Lemma 5,

$$T^{-1/2}\sum_{t=1}^{T}\left\|\left.\frac{\partial l_{u,t}(\boldsymbol{\psi})}{\partial \boldsymbol{\psi}}\right|_{\boldsymbol{\psi}_\mathbf{0}} - \left.\frac{\partial l_t(\boldsymbol{\psi})}{\partial \boldsymbol{\psi}}\right|_{\boldsymbol{\psi}_\mathbf{0}}\right\| \xrightarrow{a.s.} \mathbf{0}.$$

Thus,

$$T^{-1/2}\sum_{t=1}^{T}\frac{\partial l_t(\boldsymbol{\psi})}{\partial \boldsymbol{\psi}}\bigg|_{\boldsymbol{\psi}_0} \xrightarrow{d} \mathsf{N}(0,\boldsymbol{B}_0).$$

This completes the proof. ■

References

BACON, D. W., AND D. G. WATTS (1971): "Estimating the Transition Between Two Intersecting Lines," *Biometrika*, 58, 525–534.

BERKES, I., L. HORVÁTH, AND P. KOKOSZKA (2003): "GARCH Processes: Structure and Estimation," *Bernoulli*, 9, 201–227.

BOLLERSLEV, T. (1986): "Generalized Autoregressive Conditional Heteroskedasticity," *Journal of Econometrics*, 21, 307–328.

BOUGEROL, P., AND N. M. PICARD (1992): "Stationarity of GARCH Processes and of Some Non-negative Time Series," *Journal of Econometrics*, 52, 115–127.

BOUSSAMA, F. (2000): "Normalité Asymptotique de L´Estimateur du Pseudo-Maximum de Vraisemblance dún Modèle GARCH," *Comptes Rendus de l´Academie des Sciences, Série I*, 331, 81–84.

CHAN, F., AND M. MCALEER (2002): "Maximum Likelihood Estimation of STAR and STAR-GARCH Models: A Monte Carlo Analysis," *Journal of Applied Econometrics*, 17, 509–534.

CHAN, K. S., J. D. PETRUCCELLI, H. TONG, AND S. W. WOOLFORD (1985): "A Multiple-Threshold AR(1) Model," *Journal of Applied Probability*, 22(2), 267–279.

CHAN, K. S., AND H. TONG (1986): "On Estimating Thresholds in Autoregressive Models," *Journal of Time Series Analysis*, 7, 179–190.

CHEN, R., AND R. S. TSAY (1993): "Functional Coefficient AutoRegressive Models," *Journal of the American Statistical Association*, 88, 298–308.

CLINE, D., AND H. PU (1999a): "Geometric Ergodicity of Nonlinear Time Series," *Statistica Sinica*, 9, 1103–1118.

——— (1999b): "Stability of Nonlinear AR(1) Time Series with Delay," *Stochastic Processes and their Applications*, 82, 307–333.

COMTE, F., AND O. LIEBERMAN (2003): "Asymptotic Theory for Multivariate GARCH Processes," *Journal of Multivariate Analysis*, 84, 61–84.

ENGLE, R., AND K. KRONER (1995): "Multivariate Simultaneous Generalized ARCH," *Econometric Theory*, 11, 122–150.

ENGLE, R. F. (1982): "Autoregressive Conditional Heteroskedasticity with Estimates of the Variance of UK Inflations," *Econometrica*, 50, 987–1007.

FERRANTE, M., G. FONSECA, AND P. VIDONI (2003): "Geometric Ergodicity, Regularity of the Invariant Distribution and Inference for a Threshold Bilinear Markov Process," *Statistica Sinica*, 13, 367–384.

FRANCQ, C., AND J.-M. ZAKOÏAN (2004): “Maximum Likelihood Estimation of Pure GARCH and ARMA-GARCH Processes,” *Bernoulli*, 10, 605–637.

GOLDFELD, S. M., AND R. QUANDT (1972): *Nonlinear Methods in Econometrics*. North Holland, Amsterdam.

HE, C., AND T. TERÄSVIRTA (1999a): “Fourth Moment Structure of the GARCH(p, q) Process,” *Econometric Theory*, 15, 824–846.

——— (1999b): “Properties of Moments of a Family of GARCH Processes,” *Journal of Econometrics*, 92, 173–192.

JEANTHEAU, T. (1998): “Strong Consistency of Estimators for Multivariate ARCH Models,” *Econometric Theory*, 14, 70–86.

LEE, S.-W., AND B. E. HANSEN (1994): “Asymptotic Theory for The GARCH(1,1) Quasi-Maximum Likelihood Estimator,” *Econometric Theory*, 10, 29–52.

LI, W. K., AND K. LAM (1995): “Modelling Asymetry in Stock Returns by a Threshold Autoregressive Conditional Heterocedastic Model,” *The Statistician*, 44, 333–341.

LING, S., AND M. MCALEER (2002a): “Necessary and Sufficient Moment Conditions for the GARCH(r, s) and Asymmetric Power GARCH(r, s) Models,” *Econometric Theory*, 18, 722–729.

——— (2002b): “Stationarity and The Existence of Moments of a Family of GARCH Processes,” *Journal of Econometrics*, 106, 109–117.

——— (2003): “Asymptotic Theory for a Vector ARMA-GARCH Model,” *Econometric Theory*, 19, 280–310.

LUMSDAINE, R. (1996): “Consistency and Asymptotic Normality of the Quasi-Maximum Likelihood Estimator in IGARCH(1,1) and Covariance Stationary GARCH(1,1) Models,” *Econometrica*, 64, 575–596.

LUUKKONEN, R., P. SAIKKONEN, AND T. TERÄSVIRTA (1988): “Testing Linearity Against Smooth Transition Autoregressive Models,” *Biometrika*, 75, 491–499.

MCALEER, M. (2005): “Automated Inference and Learning in Modeling Financial Volatility,” *Econometric Theory*, 21, 232–261.

MCALEER, M., F. CHAN, AND D. MARINOVA (2002): “An Econometric Analysis of Asymmetric Volatility: Theory and Application to Patents,” *Journal of Econometrics*, in press.

MEDEIROS, M. C., AND A. VEIGA (2000): “A Hybrid Linear-Neural Model for Time Series Forecasting,” *IEEE Transactions on Neural Networks*, 11(6), 1402–1412.

——— (2005): “A Flexible Coefficient Smooth Transition Time Series Model,” *IEEE Transactions on Neural Networks*, 16, 97–113.

MIRA, S., AND A. ESCRIBANO (2000): “Nonlinear Time Series Models: Consistency and Asymptotic Normality of NLS Under New Conditions,” in *Nonlinear Econometric Modeling in Time Series Analysis*, ed. by W. A. Barnett, D. Hendry, S. Hylleberg, T. Teräsvirta, D. Tjøsthein, and A. Würtz, pp. 119–164. Cambridge University Press.

NELSON, D. B. (1990): “Stationarity and Persistence in the GARCH(1,1) Model,” *Econometric Theory*, 6, 318–334.

NEWEY, W., AND D. MCFADDEN (1994): “Large Sample Estimation and Hypothesis Testing,” in *Handbook of Econometrics*, ed. by R. F. Engle, and D. L. McFadden, vol. 4, pp. 2111–2245. Elsevier Science.

STOUT, W. F. (1974): *Almost Sure Convergence*. Academic Press, New York.

SUAREZ-FARIÑAS, M., C. PEDREIRA, AND M. MEDEIROS (2004): “Local-Global Neural Network: A New Approach for Nonlinear Time Series Modeling,” *Journal of the American Statistical Association*, 99, 1092–1107.

TERÄSVIRTA, T. (1994): “Specification, Estimation, and Evaluation of Smooth Transition Autoregressive Models,” *Journal of the American Statistical Association*, 89, 208–218.

TONG, H. (1978): “On a Threshold Model,” in *Pattern Recognition and Signal Processing*, ed. by C. H. Chen, Amsterdam. Sijthoff and Noordhoff, pp. 101-141.

——— (1990): *Non-linear Time Series: A Dynamical Systems Approach*. Oxford University Press, Oxford.

TONG, H., AND K. LIM (1980): “Threshold Autoregression, Limit Cycles and Cyclical Data (with Discussion),” *Journal of the Royal Statistical Society, Series B*, 42, 245–292.

TWEEDIE, R. (1988): “Invariant Measure for Markov Chains with No Irreducibility Assumptions,” *Journal of Applied Probability*, 25A, 275–285.

VAN DIJK, D., T. TERÄSVIRTA, AND P. FRANSES (2002): “Smooth Transition Autoregressive Models – A Survey of Recent Developments,” *Econometric Reviews*, 21, 1–47.

WHITE, H. (1994): *Estimation, Inference and Specification Analysis*. Cambridge University Press, New York, NY.

WOOLDRIDGE, J. M. (1994): “Estimation and Inference for Dependent Process,” in *Handbook of Econometrics*, ed. by R. F. Engle, and D. L. McFadden, vol. 4, pp. 2639–2738. Elsevier Science.

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