Pontifícia Universidade Católica
do Rio de Janeiro

Matheus de Barros Santa Lucci e Silva

Demographics and the Fisher Effect in the Nineteenth Century

Dissertação de Mestrado

Dissertation presented to the Programa de Pós–graduação em Economia of PUC-Rio in partial fulfillment of the requirements for the degree of Mestre em Economia.

Advisor: Prof. Carlos Viana de Carvalho

Rio de Janeiro
March 2017

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Abstract

Silva, Matheus de Barros Santa Lucci e; Carvalho, Carlos Viana de (Advisor). **Demographics and the Fisher Effect in the Nineteenth Century**. Rio de Janeiro, 2017. 51p. Dissertação de Mestrado – Departamento de Economia, Pontifícia Universidade Católica do Rio de Janeiro.

There is little response of nominal interest rates to inflationary movements in the second half of the Nineteenth Century, while the Fisher equation would predict a one-to-one relation between these economic variables. Most of the previous answers to this observation rely on some sort of irrationality argument (Fisher (1906), Friedman and Schwartz (1982), Summers (1983) and Barsky and De Long (1991) are some examples) or state that there are problems in the data used (Perez and Siegler (2003)). In this thesis, I argue that this is not due to agent irrationality, but to the lowering of the equilibrium interest rate level as a response to a demographic transition attributed to advances in medical science and enhancements in sanitation infrastructure. I build an stylized overlapping generations model based on Gertler (1999) that captures the main features of the American Economy during this period, then calibrate it and conduct experiments to show that Barsky and De Long's (1991) "strike" on the Fisher Effect does not hold when the demographic channel is turned off.

Keywords

Demographic transition; Equilibrium interest rates; Macroeconomics; Aging;

Resumo

Silva, Matheus de Barros Santa Lucci e; Carvalho, Carlos Viana de. **Demografia e Efeito Fisher no Século XIX**. Rio de Janeiro, 2017. 51p. Dissertação de Mestrado – Departamento de Economia, Pontifícia Universidade Católica do Rio de Janeiro.

Há pouca resposta das taxas nominais de juros ao movimentos da inflação na segunda metade do Século XIX, enquanto a equação de Fisher prevê uma relação de um para um da taxa nominal de juros à inflação. A maior parte das respostas a essa observação dependem, de algum jeito, de argumentos sobre a irracionalidade dos agentes econômicos (Fisher (1906), Friedman e Schwartz (1982), Summers (1983) e Barsky e De Long (1991), por exemplo), ou argumentam que os dados desse período são falhos (Perez e Siegler (2003)). Nessa dissertação, eu argumento que a taxa de juros nominal não aumentou o quanto deveria não por irracionalidade dos agentes, mas sim porque a taxa natural de juros abaixou como resposta a uma transição demográfica nesse período, atribuída às melhoras na infraestrutura de saúde pública e a avanços na ciência médica. Eu construo um modelo de gerações imbricadas estilizado com base em Gertler (1999) que captura algumas das principais características da economia americana desse período. Então, calibro-o e conduzo experimentos demográficos para mostrar que o principal argumento de Barsky e De Long (1991) contra o efeito Fisher não prossegue caso se cancelem os efeitos da transição demográfica.

Palavras-chave

Transição demográfica; Taxas de juros de equilíbrio; Macroeconomia; Envelhecimento;

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1
Introduction

During the second half of the Nineteenth Century, the discovery of new gold mines and the enhancement of mining technologies made the world gold supply grow at rates previously seen during the Gold Rush. Under the classical Gold Standard, this means the monetary stock and the price level grew at a very fast pace. The following graph shows how much the monetary stock and the price levels grew.

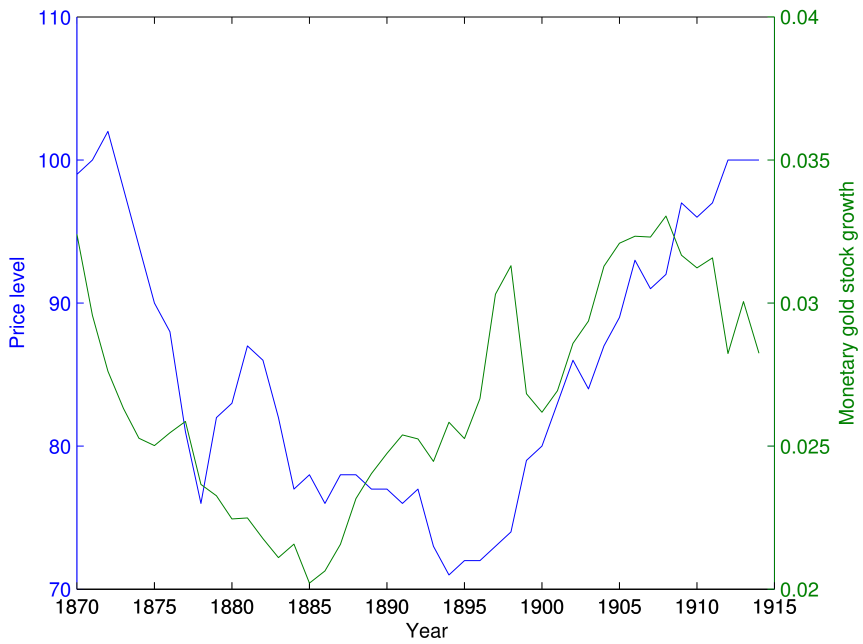

Figure 1.1: Price level and monetary gold stock growth. Source: Gold Council, Friedman and Schwartz (1982)

Despite the positive inflation after 1896, there is little response of the nominal interest rates. This is unexpected if one is to take the predction that the nominal interest rate should raise one-to-one with inflation, as stated by the Fisher equation. Figure 1.2 presents the relatively stable path of the nominal interest rate during this period.

There are two main explanations for this unresponsiveness: it is either caused by some sort of irrationality, or by problems in the data used. The first point has been made differently by many authors: Fisher (1906) argues that

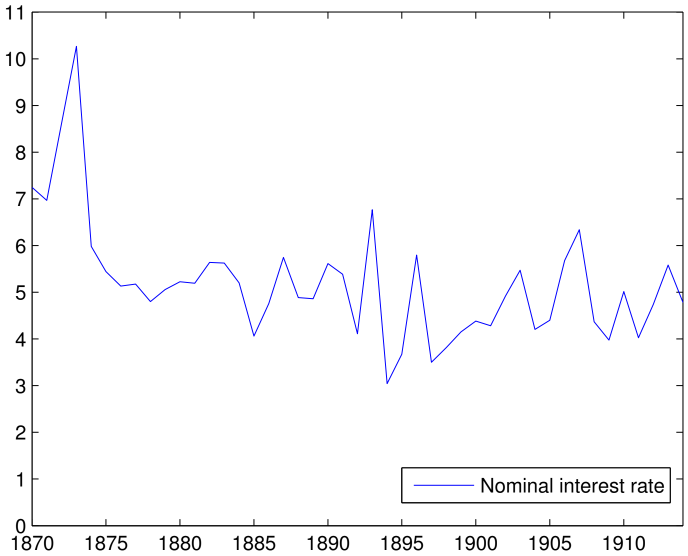

Figure 1.2: Nominal interest rate. Source: Macaulay (1938)

the gold boom must have "caught merchants innatent"; Friedman and Schwartz (1982) state that this is due to agents lagging in forming their expectations; Summers (1983) argues that agents suffered from monetary illusion; Barsky and DeLong (1991) argue that this is due to agents not being certain about the model used to analyse the economic variables. On the other hand, Perez and Siegler (2003) create a new price index with previously neglected data and show that the nominal interest rates are correlated with the expected inflation of this new price index.

In this paper, I present another explanation for the unresponsiveness of the interest rates: the demograhic changes during this period made the equilibrium real interest rate drop enough to make the nominal interest rate relatively stable during the high inflation period.

I conduct my analysis using a stylised life cycle monetary economy model built on Gertler (1999) and calibrated to the American economy over the second half of the nineteenth century up to the World War I. Other papers that use models similar to this one are Favero (2010), Carvalho and Ferrero (2014), Carvalho, Ferrero and Nechio (2016) and Kara and Von Thadden (2016). I dialog directly with Barsky and DeLong (1991) by running instrumental variables regressions and showing that the Fisher Effect hold (that is, the nominal interest rates respond to the liquidity shocks as expected) when I turn off the demographical mechanisms.

I'm dividing this paper into four other chapters beyond this Introduction. Chapter 2 presents evidences of demographic changes in the second half of the Nineteenth Century United States. I present the model in Chapters 3. Chapter 4 is devoted to presenting the calibration strategy, its results and the conclusions from the model experiments. Finally, Chapter 5 summarizes and concludes this paper. The Appendix presents the model derivation in depth.

2
Demographic Changes in the Nineteenth Century United States

I put emphasys on two demographic movements that happened during the second half of the Nineteenth Century. These are the lengthening of the average life span and the increase in the retirement period.

From 1850 to 1920, the average life duration went from 46.35 to 59.60 years (Lee, 2007). These extra years were possible due to governmental measures concerning public health and to advances in Medical Science. The reader will notice that it does not matter in the model I present where these improvements come from, so I abstain from diving deeper into discussions on whether advances in knowledge or the improvements in public health were the most significant to the increase in life expectations (McKeown, 1976).

Lemuel Shattuck is one of the biggest names in governmental interference in public health. In the beginning of 1840, he instituted the first recording system for births, marriages and deaths in the state of Massachusetts. Later in the decade, Shattuck et al (1849) present an extense survey on the sanitary conditions of Massachusetts' counties and end up by proposing fifty measures to improve them, ranging from teaching basic sanitary science to parents to the removal of inhabitants living in overcrowded lodging-houses and cellar-dwellings.

The construction of central water distribution systems has also become a priority in some cities. Urban areas also tended to have very low life quality standards, due to being densely inhabited and poverty-stricken; therefore, the creation of these distribution systems implied rapid decline of infectious diseases and mortality. The most iconic example is Chicago: in 1855, the city's Board of Sewerage Commissioners ordered the raise of the street level. Over the next decades, the city has been literally lifted out of the Lake Michigan level, lowering the number of water overflows in the city and the number of waterborne diseases.

The contributions of the advances in Medical Science come from two fronts, the first one being the theories on the origin of the diseases. Pasteur proposed and presented convincing evidence of the Germ Theory in the first years of the 1860s. Over the subsequent decades, the theory was improved and advanced by the work of Koch and others and began to be accepted by a then very conservative medical profession. Another important work is Budd's (1873) treatise on typhoid fever, which has led doctors and scientists to accept diseases could be transmitted by contagion[1].

Besides these theoretical contributions on the origin of the diseases, vaccines and new techniques were created and improved. Even though the first vaccine was invented in the end of the Eighteenth Century, it took almost 100 years until new vaccines appeared. In 1879, Pasteur created the cholera vaccine by weakening its bacteria and injecting them in chicken. In 1885, Pasteur saved the life of a boy infected by rabies by applying a series of 13 shots of weakened rabies virus. The vaccines for tetanus, typhoid fever and the bubonic plague came soon after. Another important new procedure is the laryx intubation. It was invented in during the 1880s and improved in 1888, this later version of the procedure was quickly adopted by doctors worldwide, saving many lifes that would have been previously threated by diseases that inflict damage or completely close the respiratory ways.

Figure 2.1 presents the death rates (per 100,000) of different diseases in Massachusetts[2] over the period of interest. There is an overall decline in the mortality rates of diseases caused by bacteria and viruses, despite a few sporadic outbreaks. There are two features of this data that are worth noticing. The first one is the great fall in tuberculosis' death rate. Tuberculosis is an airborne disease whose vaccine was only invented after World War I, its decline is therefore attributed to general improvements in sanitary conditions. The second important feature of this data is the increase steady increase in cancer[3] death rates.Since cancer is typically associated with the elderly (Doll, 1971), this is another evidence of the American population getting older.

[1]Reviews and notices. BMJ 1874;ii: 835-8. Commenting on Typhoid Fever: Its Nature, Mode Of Spreading, And Prevention. By William Budd, MD FRS. London: Longmans, Green, and Co, 1873

[2]Massachussets isn't representative of the other states because it has a much higher urbanization rate and income than the rest of the states. It's worth remembering that in this period cities were the home of the poor, and this is correlated to bad sanitary, alimentary and working conditions

[3]Under the broad label of 'Malignant neoplasms'

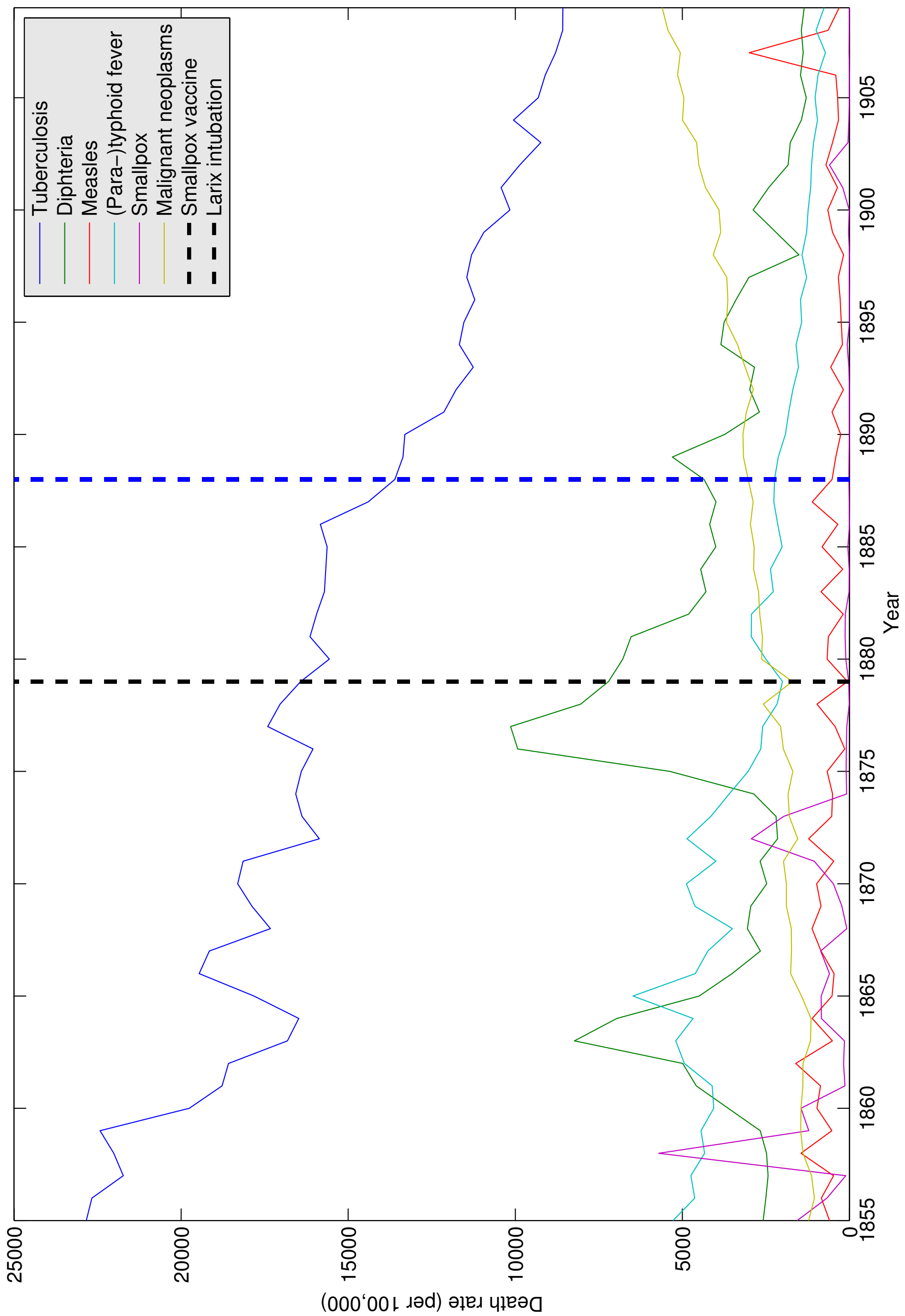

Figure 2.1: Death rates by causes in nineteenth century Massachussets. Source: Compiled in Carter et al (2006)

Newspapers citations provide some anedoctal evidence of the increased concerns about public health issues. Figure 2.2 presents citations of selected words related to these topics in newspapers extracted from Elephind, a free-to-access historical newspaper search engine that searches over 2,500,000 historical American newspapers. The usage of the word "epidemic", for example, grew quickly after the 1870s, and "mortality"exhibited a steady growth throughout the whole period. The timing of these increases follow closely those of the medical discoveries and improvements.

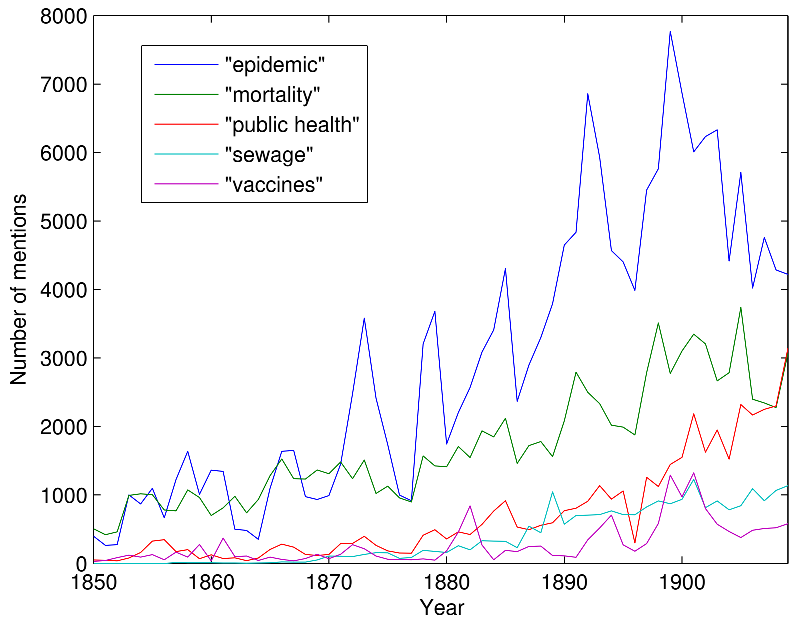

Figure 2.2: Newspaper citations of selected words. Source: Elephind

All this previous evidence is backed by Censuses data. Haines (1994) constructed the life tables for the United States based on the Census held decenially and, based on his estimates, I constructed the following year-of-birth cohort age-t radix presented in Figure 2.3. This measure shows how many individuals (out of a standardized initial population of 100,000) survive throughout the years. For illustration purposes, the starting age is set to 10. The outwards movements in the function reflects the increase in life span, since it means that more people are surviving between periods. Table 2.1 presents the average life lenght, as estimated by Lee (2007).

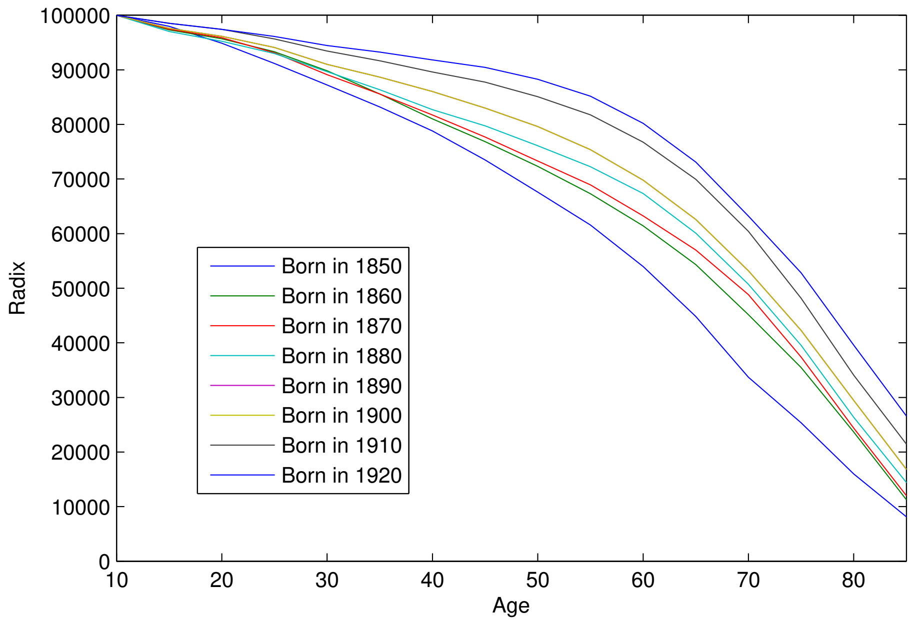

Figure 2.3: Cohort radix function. Source: Own calculations, data comes from Haines (1994)

	Estimate
1850	46.35
1860	47.11
1870	47.53
1880	49.78
1890	51.28
1900	52.79
1910	55.81
1920	59.60

Source: Lee (2007)

The next question is: "how are these extra years of life distributed between work and retirement?"This is a fundamental point in the model because it dictates whether individuals will save more for retirement or less. At first glance, one could expect agents would simply work more until death. Ransom and Sutch (1986) argue that even though this assumption is common in the Economic History literature, retirement in the second half of the nineteenth century exists and is significant. Lee (2007) provides numerical estimates for these movements inside cohorts. His estimates point that both working and retirement time increased during this period, with retirement time increasing relatively more than the increase in working time for 20 year old individuals.

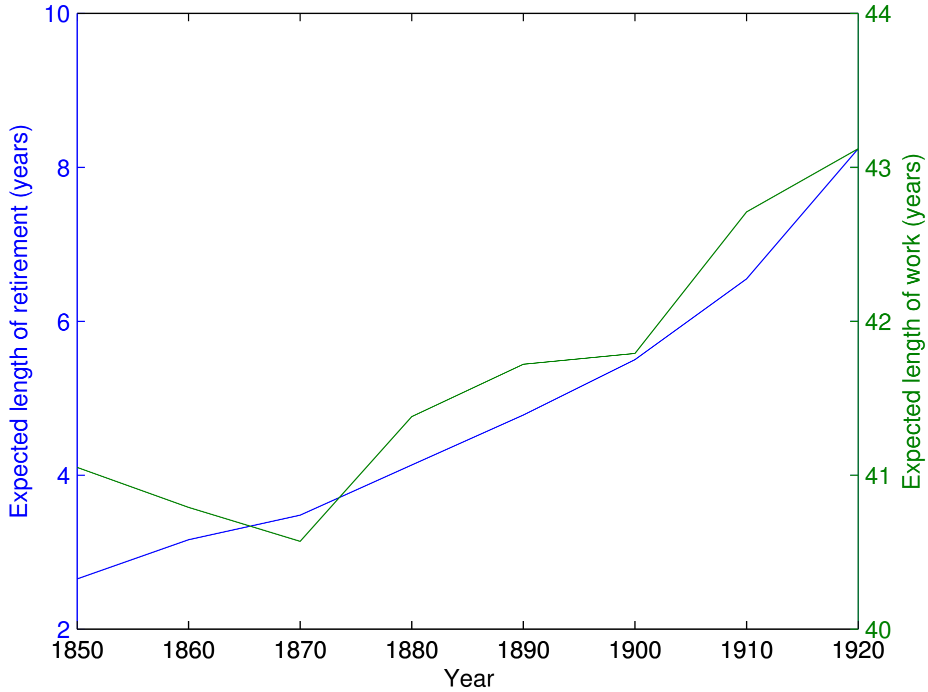

Figure 2.4: Estimated length of work and retirement. Source: Lee (2007)

3
Model

I build the model on Gertler's (1999) overlapping generations model. The setup includes two economic actors: individuals and firms. There are two types of individuals: workers (who inelastically supply one unit of labor at every period), and retirees (who can only consume out of their assets). All individuals are born workers and deprived of capital. At the beginning of every period a worker faces the probability $1 - \omega_t$ of becoming a retiree and ω_t of staying a worker. Once retired, the death clock begins to tick, and a retiree now faces the probability γ_t of dying at the beginning of period t. For simplicity, retired individuals cannot go back into the labor market. These probabilities imply that the average time employed at time t is $(1 - \omega_t)^{-1}$, and that the expected retirement at time t is $(1 - \gamma_t)^{-1}$. This dynamics is easily summarized in the following diagram:

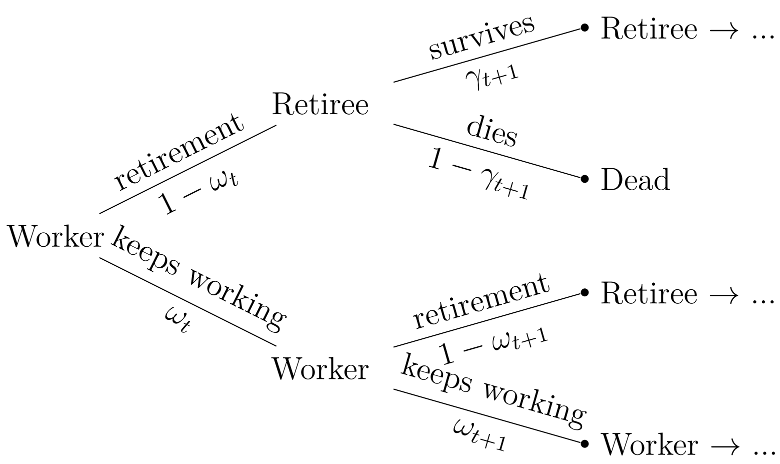

Figure 3.1: Diagram ilustrating the demographic evolution. Source: Own elaboration

Labor force N_t^w and the number of retirees evolve according to:

$$N_t^w = (1 - \omega_t + n_t)N_{t-1}^w + \omega_t N_{t-1}^w = (1 + n_t)N_{t-1}^w$$

and

$$N_t^r = (1 - \omega_t)N_{t-1}^w + \gamma_t N_{t-1}^r$$

The dependency ratio $\vartheta_t \equiv \frac{N_t^r}{N_t^w}$ evolves according to:

$$(1+n_t)\vartheta_t = (1-\omega_t) + \gamma_t \vartheta_{t-1}$$

I am not modelling aggregate uncertainty in this framework, so that individuals face only two idiosyncratic risks: workers suddenly losing their income and retirees not knowing the time of their death. The uncertainty of dying once retired is counterbalanced by a perfect annuities market in which all (and only) retirees take part of. Each retiree hands their assets to this mutual fund and those who survive the death lottery (that is $\gamma_t N_{t-1}^r$, receive all the returns R_t/γ_t.

The uncertainty faced by workers is addressed by using a CES nonexpected utility function, as in Epstein-Zin (1989) or Weil (1990). In this setup, individuals are risk neutral with respect to the risk of suddenly losing income due to retirement, but it still allows for an arbitrary intertemporal elasticity of substitution. As Gertler (1999) states, it is desirable to mitigate the impact of the retirement lottery, once it derives straight from the artificial retirement scheme in this model. The parameter ρ determines the intertemporal elasticity of consumption, given by $\sigma \equiv (1-\rho)^{-1}$.

I model the Gold Standard period by including gold as a commodity money, denoted by M. This modelling strategy ought to determine the nominal side of this economy, where resides the main interest of this paper. In the setup presented here, the real money holdings enter in the utility multiplied by a constant μ, later calibrated. I denote V_t^z as the utility of an individual z, $z \in \{w(orker), r(etiree)\}$, and C_t^z and M_t^z his consumption and nominal money demand at time t, respectively.

Preferences are then given by:

$$V_t^z = \left\{ (C_t^z)^\rho + \mu\left(\frac{M_t^z}{P_t}\right)^\rho + \beta_{t+1}^z \mathbb{E}_t[V_{t+1}|z]^\rho \right\}^{\frac{1}{\rho}}$$

Where

$$\beta_{t+1}^z = \begin{cases} \beta & , if \text{z=w} \\ \beta\gamma_{t+1} & , if \text{z=r} \end{cases}$$

$$\mathbb{E}_t[V_{t+1}|z] = \begin{cases} (1-\omega_{t+1})V_t^r + \omega_{t+1}V_t^w & , if \text{z=w} \\ V_t^r & , if \text{z=r} \end{cases}$$

Individuals choose not only the amount of consumption they will enjoy at period t, C_t^z, but also the monetary holdings M_t^z, the amount of capital they will save, K_t^z and the amount of firm shares x_{Ft}^z.

3.1 Household problems

In this section I use the general setup provided to solve for both retiree and worker problems. I later aggregate both.

3.1.1 The retiree's problem

The optimization problem of a retiree born at time j and retired at time τ can be recursively written as:

$$V_t^r(j,\tau) = \max\left\{(C_t^r(j,\tau))^\rho + \mu\left(\frac{M_t^r(j,\tau)}{P_t}\right)^\rho + \beta\gamma_{t+1}(V_{t+1}^r(j,\tau))^\rho\right\}^{\frac{1}{\rho}}$$

subject to

$$C_t^r(j,\tau) \quad + \quad \frac{M_t^r(j,\tau)}{P_t} + K_t^r(j,\tau) + \frac{B_t^r(j,\tau)}{P_t} + \frac{P_t^F}{P_t} x_{Ft}^r(j,\tau) =$$

$$= \frac{1}{\gamma_t}\left\{\left[R_t^K + (1-\delta)\right] K_{t-1}^r(j,\tau) + \frac{\left(P_t^F + D_t^F\right)}{P_t} x_{F,t-1}^r(j,\tau) + \frac{1}{\pi_t}\frac{M_{t-1}^r(j,\tau)}{P_{t-1}}\right\}, \; \forall t$$

Where M_t^r/P_t are the real monetary holdings, R_t^K is the rate of return of capital, P_{Ft} is the price of firm shares, π_t is the gross consumption inflation rate $\pi_t \equiv P_t/P_{t-1}$ and D_{Ft} are the dividends distributed by the firm. The solution to this problems yields an Euler Equation,

$$\frac{(C_t^r(j,\tau))^{\rho-1}}{(C_{t+1}^r(j,\tau))^{\rho-1}} = \beta\left[R_{t+1}^K + (1-\delta)\right]$$

A no arbitrage condition,

$$\frac{P_{t+1}^F + D_{t+1}^F}{P_{Ft}} = \frac{R_t}{\pi_{t+1}} = R_{t+1}^K + (1-\delta)$$

And a relationship between monetary holdings and consumption,

$$\frac{M_t^r(j,\tau)}{P_t} = C_t^r(j,\tau)\left\{\frac{\mu}{1-1/R_{t+1}}\right\}^\sigma$$

To solve the model, let us define the total retiree's assets as:

$$A_t^r(j,\tau) \equiv K_t^r(j,\tau) + \frac{B_t^r(j,\tau)}{P_t} + \frac{P_t^F}{P_t} x_{F,t}^r(j,\tau) + \frac{1}{R_{t+1}} \frac{M_t^r(j,\tau)}{P_t}$$

Then, I guess and verify the following solution:

$$C_t^r(j,\tau) + (1 - 1/R_{t+1}) \frac{M_t^r(j,\tau)}{P_t} = \epsilon_t \xi_t \left(\frac{1}{\gamma_t} \frac{R_{t-1}}{\pi_t} A_{t-1}^r(j,\tau) \right)$$

Where $\epsilon_t \xi_t$ is the retiree's marginal propensity to consume out of assets. I follow Gertler (1999) in defining the worker's marginal propensity to consume out of wealth as ξ_t because the ration between these propensities, ϵ_t, is important in the model. The non-linear difference equation that governs this value is:

$$1 - \Psi_{t+1} \frac{\epsilon_t \xi_t}{\epsilon_{t+1} \xi_{t+1}} \gamma_{t+1} \beta^\sigma \left(\frac{R_t}{\pi_{t+1}} \right)^{\sigma-1} = \epsilon_t \xi_t$$

Where,

$$\Psi_{t+1} \equiv \frac{\left[1 + \mu^\sigma \left\{ \frac{R_t}{R_t - 1} \right\}^{\sigma-1} \right]^{-1}}{\left[1 + \mu^\sigma \left\{ \frac{R_{t+1}}{R_{t+1} - 1} \right\}^{\sigma-1} \right]^{-1}}$$

Given the linear forms of the solutions and the fact that they do not depend on individual characteristics of the retirees, it is straightforward that the money demand by retirees is given by:

$$\frac{M_t^r}{P_t} = C_t^r \left\{ \frac{\mu}{1 - 1/R_{t+1}} \right\}^\sigma$$

Plugging this into the solution guess, one obtains the closed form solution of the aggregate amount retirees consume at the current period:

$$C_t^r \left(1 + (1 - 1/R_{t+1})^{1-\sigma} \mu^\sigma \right) = \epsilon_t \xi_t \left(\frac{1}{\gamma_t} \frac{R_{t-1}}{\pi_t} A_{t-1}^r \right)$$

3.1.2
The worker's problem

The solution to the worker problem follows basically the same steps as the one for the retirees. To make the solution easier, I have defined the worker's assets A_t^w analogously to the retiree's assets, done previously. The utility now takes into account the fact the worker may retire next period.

The recursive formulation of the optmization problem solved by the worker born at time j is:

$$V_t^w(j) = \max \left\{ (C_t^w(j))^\rho + \mu \left(\frac{M_t^w(j)}{P_t} \right)^\rho + \beta \left[\omega_{t+1} V_{t+1}^w(j) + (1 - \omega_{t+1}) V_{t+1}^r(j) \right]^\rho \right\}^{\frac{1}{\rho}}$$

subject to

$$C_t^w(j) + (1 - 1/R_{t+1}) \frac{M_t^w(j)}{P_t} + A_t^w(j) = W_t - T_t^w + \frac{R_{t-1}}{\pi_t} A_{t-1}^r(j)$$

The Euler equation is:

$$C_t^w(j) \left[\beta \Omega_{t+1} \frac{R_t}{\pi_{t+1}} \right]^{\frac{1}{1-\rho}} = \omega_{t+1} C_{t+1}^w(j, t+1) + (1 - \omega_{t+1}) \left(\frac{\Delta_{t+1}^r}{\Delta_{t+1}^w} \right) C_{t+1}^r(j)$$

Where Ω_t is a return adjustment term defined as:

$$\Omega_t = \omega_t + (1 - \omega_t)(\epsilon_t)^{(1/(1-\sigma))}$$

Money demand function of the worker has the same form as before:

$$\frac{M_t^w(j)}{P_t} = C_t^w(j) \left[\frac{\mu}{1 - 1/R_{t+1}} \right]^\sigma$$

To solve the problem, I guess the following solution for workers who are still in labor force:

$$C_t^w(j) + (1 - 1/R_{t+1}) \frac{M_t^w(j)}{P_t} = \xi_t \left(\frac{R_{t-1}}{\pi_t} A_{t-1}^w(j) + H_t^w \right)$$

And for those who just left it,

$$C_t^r(j) + (1 - 1/R_{t+1}) \frac{M_t^r(j)}{P_t} = \epsilon_t \xi_t \left(\frac{R_{t-1}}{\pi_t} A_{t-1}^w(j) \right)$$

Where H_t^w is the discounted sum of all non-financial wealth, defined as:

$$H_t^w \equiv \sum_{\nu=0}^{\infty} \frac{\left(\frac{W_{t+\nu}}{P_{t+\nu}} \right)}{\Pi_{s=1}^{\nu} \left[\frac{\Omega_{t+s} R_{t+s-1}}{\omega_{t+s} \pi_{t+s}} \right]} = \frac{W_t}{P_t} + \frac{\omega_{t+1} \pi_{t+1} H_{t+1}^w}{\Omega_t R_t}$$

Taking the same steps to follow the retiree's problem, I obtain the difference equation that governs the marginal propensity of the worker to consume out of assets,

$$\xi_t = 1 - \frac{\xi_t}{\xi_{t+1}} \beta^\sigma \left[\Omega_{t+1} \frac{R_t}{\pi_{t+1}} \right]^{\sigma-1} \Psi_{t+1}$$

Since these solutions are linear and do not depend on the period that the agent is born, obtaining the closed form solutions for the aggregate quantities demanded by workers is simple. The workers's money demand is:

$$\frac{M_t^w}{P_t} = C_t^w \left\{ \frac{\mu}{1 - 1/R_{t+1}} \right\}^\sigma$$

And the closed form solution of the aggregate amount workers consume at the current period:

$$C_t^w \left(1 + (1 - 1/R_{t+1})^{1-\sigma} \mu^\sigma \right) = \epsilon_t \xi_t \left(\frac{1}{\gamma_t} \frac{R_{t-1}}{\pi_t} A_{t-1}^w \right)$$

3.1.3
Household Aggregation

Evolution of the assets for each cohort is easily checked with budget restrictions. Let

$$\lambda_t \equiv \frac{A_t^r}{A_t}$$

Then aggregate consumption is given by the sum of the consumption solutions of retirees and workers:

$$C_t \left[1 + \mu^\sigma (1 - 1/R_{t+1})^{1-\sigma} \right] = \xi_t \left\{ \frac{R_{t-1}}{\pi_t} A_{t-1} \left[1 + \lambda_{t-1} (\epsilon_t - 1) \right] + H_t \right\}$$

Total assets evolve according to:

$$\left[\lambda_t - (1 - \omega_{t+1}) \right] A_t = \omega_{t+1} (1 - \xi_t^r) \lambda_{t-1} \frac{R_{t-1} A_{t-1}}{\pi_t^c}$$

As for the total money demand,

$$\frac{M_t^d}{P_t} = C_t \left[\frac{\mu}{1 - 1/R_{t+1}} \right]^\sigma$$

3.2
Firms

There is a representative firm operating in a perfectly competitive market, and solves the following profit maximization problem:

$$\max_{N_t, K_t} P_t Y_t - R_t^k K_t - W_t N_t$$

Subject to the technological constraint,

$$Y_t = (X_t N_t)^{\alpha} K_t^{1-\alpha}$$

The first order conditions to this problem yield the gross rate of return of capital, R_t^K and the real wage W_t/P_t, given by:

$$R_t^K = (1-\alpha)\frac{Y_t}{K_t}$$

And,

$$\frac{W_t}{P_t} = \alpha \frac{Y_t}{N_t}$$

3.3 Equilibrium

The equilibrium in this economy is defined as:

Definition 1 (Equilibrium) *Given an exogenous demographic process $\{n_t, \gamma_t, \omega_t\}_{t=1}^{\infty}$, an exogenous technological process $\{x_t\}_{t=1}^{\infty}$, and an exogenous money supply $\{M_t^s\}_{t=1}^{\infty}$, an equilibrium is a sequence of quantities $\{C_t, A_t, \lambda_t, H_t^w, Y_t, K_t, M_t^d\}_{t=1}^{\infty}$, marginal propensities to consume $\{\epsilon_t, \xi_t, \Omega_t\}_{t=1}^{\infty}$, prices $\{R_t^K, \frac{W_t}{P_t}, P_t\}_{t=1}^{\infty}$ and dependency ratios $\{\vartheta_t\}_{t=1}^{\infty}$, such that:*

a. *Both retirees and workers solve their utility maximization problem subject to their budget constraints*
b. *Final goods firm maximizes profits subject to technology, and intermediate goods firms maximize profits subject to their technological constraint and taking good demand as given*
c. *Markets clear (i.e., there is no excess demand or supply in assets, goods and monetary markets)*

3.4 Demographic impacts on variables

Changes in demographics, specially the increased life expectancy, will force down the real interest rates because both workers and retirees will save more in order to self-finance their consumption in the extra retirement years. Taking the partial derivative of the retiree's marginal propensity to consume out of wealth,

$$\frac{\partial(\epsilon\xi)}{\partial\gamma} = -\beta^{\sigma} R^{\sigma-1}\left[1 + (\sigma - 1)\,\epsilon_{R,\gamma}\right] < 0$$

Where $\epsilon_{R,\gamma}$ (< 0) is the elasticity of the real interest rate with respect to the probability of surviving once retire, that is, $\epsilon_{R,\gamma} \equiv (\partial R/R) / (\partial\gamma/\gamma)$.

The implicit function theorem yields the same signal for $\frac{\partial\xi}{\partial\gamma}$:

$$\frac{\partial\xi}{\partial\gamma} = -\frac{\beta^{\sigma}(\Omega R)^{\sigma-1}\left[(\sigma-1)\,\epsilon_{R,\gamma} - \left(\frac{\Omega-\omega}{\Omega}\right)\epsilon_{\epsilon\xi,\gamma}\right]}{\gamma\left[1+\beta^{\sigma}(\Omega R)^{\sigma-1}\left(\frac{\Omega-\omega}{\Omega}\right)\frac{1}{\epsilon\xi}\right]} < 0$$

Where $\epsilon_{\epsilon\xi,\gamma} \equiv \epsilon_{R,\gamma} \equiv (\partial\epsilon\xi/\epsilon\xi) / (\partial\gamma/\gamma)$ (this is negative because $\partial(\epsilon\xi)/\partial\gamma$ is negative).

4
Calibration and Model experiments

I calibrate the model to mimic the movements of the economic variables of the American economy between the years of 1850 and 1920. Each period is set to one year. Agents are "born" in the model when they are 20 years old, because at this age nearly all individuals are in the workforce [1]. The demographic transition is driven by the changes in life expectation (both γ_t and ω_t) and changes in the growth of the labor force, n_t. I use Lee's (2007) estimates to obtain values for γ_t and ω_t, and then interpolate them to get the yearly series. Labor force growth is calibrated using basically the same process: I interpolate U.S. Census population estimates to get yearly data and then calculate its growth.

To calibrate the depreciation rate, δ, I use the net capital value and the value of machinery and structures purchases available in Carter et al (2006) to obtain the value of 15%.

My calibration of the technological process is trickier and erratic. To begin with, I calibrate it based only on manufactures, while it should be taken into account that the farming sector is representative. For the years 1866 to 1914, I use Frickey's (1947) manufacturing production index as my output measure, a manufacturing capital stock measure (Carter et al (2006)) and the interpolated population to calculate the technological growth according to usual growth accounting. To calibrate technological long-run growth, I follow Tamura et al (2013) and set $x = 0.01$. As a robustness exercise, I also provide the results by setting $x = 0.01$ for all periods.

The world gold stock data is obtained from the Gold Council. Finally, the remaining parameters (that is, the discount rate β, the real money balances preference parameter μ and the intertemporal elasticity of substitution σ) are calibrated by minimizing the distance from the generated interest and inflation time series with respect the the observed ones:

[1] There is indeed a very high child labor participation in workforce during the Nineteenth Century. However, the share of less-than-20-years in workforce (in comparison to their own cohort) fluctuates around 70%. Setting this age is also convenient because it matches the evidence presented in Chapter 2.

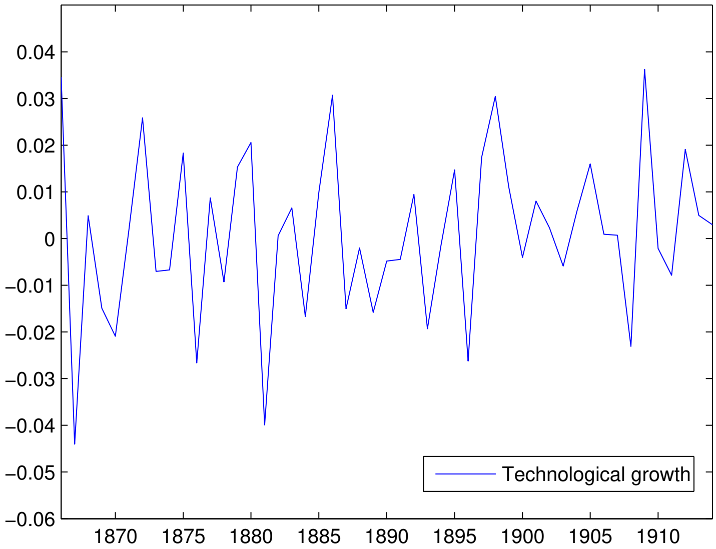

Figure 4.1: Calibration of technological growth. Source: Own calculations, data comes from Frickey (1947) and Carter et al (2006)

$$\min_{\{\beta,\mu,\sigma\}} \sum_{t=1870}^{1914} (R_t^{model}(\beta,\mu,\sigma) - R_t^{data})^2 + (\pi_t^{model}(\beta,\mu,\sigma) - \pi_t^{data})^2$$

The baseline calibration parameter values are summarized in the table below:

Parameter	Value	Description
	Internal calibration	
β	0.96	discount factor
μ	0.40	utility parameter
σ	0.30	utility parameter
	External calibration	
x	0.01	steady-state technology growth Tamura et. al (2013)
δ	0.15	depreciation rate
n	0.0186	steady-state population growth
γ	0.8786	steady-state survival probability
ω	0.9768	steady-state probability of staying a worker

Source: Model calibration, Carter et al (2006), Lee (2007), Tamura et al (2013)

Simulating the model with the calibration above, I obtain the paths for the endogenous variables in Figure 4.2.

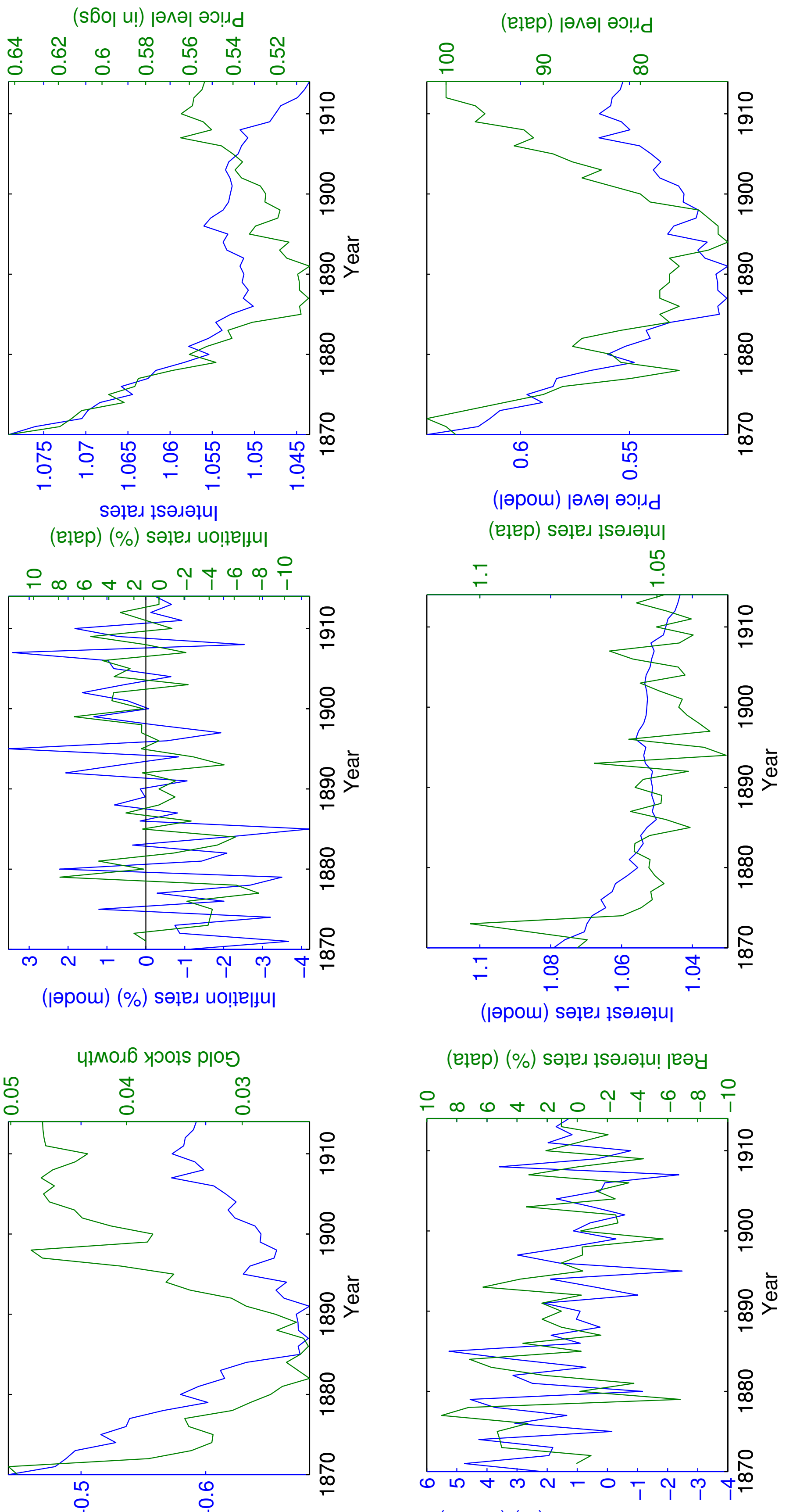

Figure 4.2: Outputs of model simulation. Source: Model estimates

Notice that the model captures well the low frequency movements of its observed counterparts. Of course, it does poorly around the 1893 crisis and in the years before World War I, for example, but these shorter frequency are not contemplated in our setup.

4.1 Model experiments

After having detailed the model and its calibration, let me recall our question: "Can demographic movements explain Barsky and DeLong's (1991) observation about the Fisher equation estimates?"

First of all, the Fisher equation says there should be a one-to-one relationship between expected inflation and nominal interest rates. Barsky and DeLong (1991) notice that this does not happen if one uses monetary stock growth as an instrument for inflation. Even during the Classical Gold Standard period. They attribute this failure due to economic agents being uncertain about the model (and its parameters) used to forecast inflation.

My model tells us a different story. While the standard rationality hypothesis still holds, movements in the natural interests rate make the econometrician see a biased parameter estimate in the regression of interest. To shed light on this point, take the equation Barsky and DeLong (1991) estimate:

$$R_t = \beta_0 + \beta_1 \pi_t + \epsilon_t$$

It's trivial that the IV estimates for β_0 and β_1 are biased if the true underlying regression is:

$$R_t = \beta_{0,t} + \beta_1 \pi_t + \epsilon_t$$

With a time-varying parameter $\beta_{0,t}$.

This is exactly the point I make in this paper: fluctuations in the equilibrium interest rate are not correctly captured by Barsky and DeLong's (1991) approach. Therefore, their point of running this IV regression is flawed when the equilibrium interest rates change over time. This shows up when I cancel the demographic movements and the expected one-to-one relation between inflation and interest rates should arises.

Table 4.2 presents the parameter estimates for the IV regressions of interest. The first row presents the estimate using the American data, while the second one uses the simulated data from the model. Notice that, even though the coefficients are not the same, the lesson is: the econometrician cannot observe the unitary coefficient.

The third row presents the IV estimate from the model-generated data if I rule out the demographic transition, but still maintaining the overlapping generation structure. That is, I set $\gamma_t = \gamma$, $\omega_t = \omega$ and $n_t = n$, and γ, ω and n are set in order to generate the mean work and retirement and population growth in the data.

The fourth row presents the estimate from the model-generated data if I set $\omega = 1$. This is a special case nested in the model, the workhorse infinetly lived agents model.

Regression	Coefficient	Standard deviation
Data	0.112	0.185
Standard calibration	-0.645	0.368
Ignoring demographic transition	3.472	1.783
Infinitely lived agents	0.908	0.398

Source: Model estimates

These calculations show that when the demographic channel is crucial to understanding what an econometrician observes. It's also worth noticing that the unitary coefficient is inside a 95% confidence interval built around the estimates for the models where the demographic transition has been turned off and where agents live indefinitely. Therefore, if a econometrician were to conduct a hypothesis test on whether the Fisher equation holds (i.e., a test of equality of the true parameter to the unit), then he would not refute this hypothesis.

4.1.1 Robustness check

Since the estimates for the technological growth are quite erratic, I also present the results for the case where I set $x = 0.01$ for all periods, following Tamura et al (2013). The estimates change, but the lessons are the same: the demographic transition is still key to understanding the unresponsiveness of the interest rates to the monetary shocks.

Regression	Coefficient	Standard deviation
Data	0.112	0.185
Standard calibration	0.679	0.329
Ignoring demographic transition	1.269	0.156
Infinitely lived agents	0.737	0.483

Source: Model estimates

5
Conclusion

In this paper, I analysed the unresponsiveness of the American nominal interest rate to the inflationary movement that happened in the second half of the Nineteenth Century with a model that takes into consideration the demographic movements that happened during this period.

I calibrate and simulate an overlapping generations model built on Gertler (1999). I then show that, when ignoring the demographic transitions in the Nineteenth Century (either by setting constant the overlapping generations structure, or by eliminating it completely with a representative infinitely lived agent model), the IV regression ran by Barsky and DeLong (1991) yield the expected unity coefficient.

To sum up, I contribute with a new explanation to that fact. Previous authors (Fisher, 1906; Friedman and Schwartz, 1982; Summers, 1983; Barsky and DeLong, 1991) have relied either on a irrationality assumption, or on an economic mismeasurement argument (Perez and Siegler, 2003). My model shows, under standard assumptions, that the demographic transition that happened during these years could have lowered the equilibrium real interest rate enough to compensate for the inflation originated from the gold mining boom.

6
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7 Appendix

This Appendix presents the model derivation in depth. First, I present the solution to the retiree's problem. Then to the worker's one. Finally, I show how to aggregate these agents to obtain the final aggregate form for the endogenous variables and the equilibrium conditions.

7.1 The retiree's problem

The retiree's problem can be recursively written as:

$$V_t^r = \max \left\{ (C_t^r)^\rho + \mu \left(\frac{M_t^r}{P_t} \right)^\rho + \beta \gamma_{t+1} \left(V_{t+1}^r \right)^\rho \right\}^{\frac{1}{\rho}}$$

subject to

$$\begin{aligned} P_t \quad C_t^r & + M_t^r + P_t K_t^r + B_t^r + P_t^F x_{Ft}^r = \\ = & \quad \frac{1}{\gamma_t} \left\{ \left[R_t^K + (1-\delta) \right] P_t K_{t-1}^r + R_{t-1} B_{t-1}^r + \left(P_t^F + D_t^F \right) x_{F,t-1}^r + M_{t-1}^r \right\} \end{aligned}$$

$$\begin{aligned} C_t^r & + \frac{1}{P_t} M_t^r + K_t^r + \frac{B_t^r}{P_t} + \frac{P_t^F}{P_t} x_{Ft}^r = \\ = & \quad \frac{1}{\gamma_t} \left\{ \left[R_t^K + (1-\delta) \right] K_{t-1}^r + \frac{R_{t-1}}{\pi_t} \frac{B_{t-1}^r}{P_{t-1}} + \frac{\left(P_t^F + D_t^F \right)}{P_t} x_{F,t-1}^r + \frac{1}{\pi_t} \frac{M_{t-1}^r}{P_{t-1}} \right\} \end{aligned}$$

Or still,

$$\begin{aligned} C_t^r & + \frac{M_t^r}{P_t} + K_t^r + \frac{B_t^r}{P_t} + \frac{P_t^F}{P_t} x_{Ft}^r = \\ = & \quad \frac{1}{\gamma_t} \left\{ \left[R_t^K + (1-\delta) \right] K_{t-1}^r + \frac{R_{t-1}}{\pi_t} \frac{B_{t-1}^r}{P_{t-1}} + \frac{\left(P_t^F + D_t^F \right)}{P_t} x_{F,t-1}^r + \frac{1}{\pi_t} \frac{M_{t-1}^r}{P_{t-1}} \right\} \end{aligned}$$

Taking the first order conditions with respect to capital, bonds, shares and real money holdings,

$$(C_t^r)^{\rho-1} = \beta \gamma_{t+1} \left(V_{t+1}^r \right)^{\rho-1} \frac{\partial V_{t+1}^r}{\partial K_t^r}$$

$$(C_t^r)^{\rho-1} = \beta\gamma_{t+1}\left(V_{t+1}^r\right)^{\rho-1}\frac{\partial V_{t+1}^r}{\partial\left(B_t^r/P_t\right)}$$

$$(C_t^r)^{\rho-1} = \frac{1}{P_{Ft}}\beta\gamma_{t+1}\left(V_{t+1}^r\right)^{\rho-1}\frac{\partial V_{t+1}^r}{\partial x_{Ft}^r}$$

$$(C_t^r)^{\rho-1} = \mu\left(\frac{M_t^r}{P_t}\right)^{\rho-1} + \beta\gamma_{t+1}\left(V_{t+1}^r\right)^{\rho-1}\frac{\partial V_{t+1}^r}{\partial\left(M_t^r/P_t\right)}$$

The envelope theorem yields the following relations:

$$\frac{\partial V_t^r}{\partial K_{t-1}^r} = (V_t^r)^{1-\rho}(C_t^r)^{\rho-1}\left[\frac{R_t^K + (1-\delta)}{\gamma_t}\right]$$

$$\frac{\partial V_t^r}{\partial\left(B_{t-1}^r/P_{t-1}^c\right)} = (V_t^r)^{1-\rho}(C_t^r)^{\rho-1}\left[\frac{1}{\gamma_t}\frac{R_{t-1}}{\pi_t^c}\right]$$

$$\frac{\partial V_t^r}{\partial K_{t-1}^r} = (V_t^r)^{1-\rho}(C_t^r)^{\rho-1}\left[\frac{1}{\gamma_t}\frac{\left(P_t^F + D_t^F\right)}{P_t^c}\right]$$

$$\frac{\partial V_t^r}{\partial\left(\frac{M_{t-1}^r}{P_{t-1}}\right)} = (V_t^r)^{1-\rho}(C_t^r)^{\rho-1}\frac{1}{\pi_t}\frac{1}{\gamma_t}$$

It follows from the first three that,

$$\frac{(C_t^r)^{\rho-1}}{(C_{t+1}^r)^{\rho-1}} = \beta\left[R_{t+1}^K + (1-\delta)\right]$$

$$\frac{(C_t^r)^{\rho-1}}{(C_{t+1}^r)^{\rho-1}} = \beta\left[\frac{R_t}{\pi_{t+1}}\right]$$

$$\frac{(C_t^r)^{\rho-1}}{(C_{t+1}^r)^{\rho-1}} = \beta\left[\frac{P_{t+1}^F + D_{t+1}^F}{P_{Ft}}\right]$$

And we end up with the no arbitrage conditions,

$$\frac{P_{t+1}^F + D_{t+1}^F}{P_{Ft}} = \frac{R_t}{\pi_{t+1}} = R_{t+1}^K + (1-\delta)$$

The real money holdings condition yields a slightly different equation,

$$(C_t^r)^{\rho-1} = \mu\left(\frac{M_t^r}{P_t}\right)^{\rho-1} + \beta\left(C_{t+1}^r\right)^{\rho-1}\frac{1}{\pi_{t+1}}$$

Merging with our first order condition w.r.t. bonds,

$$\begin{aligned}
(C_t^r)^{\rho-1} &= \mu\left(\frac{M_t^r}{P_t}\right)^{\rho-1} + \beta\gamma_{t+1}\left(V_{t+1}^r\right)^{\rho-1}\left(V_{t+1}^r\right)^{1-\rho}\left(C_{t+1}^r\right)^{\rho-1}\frac{1}{\pi_{t+1}}\frac{1}{\gamma_{t+1}} \\
(C_t^r)^{\rho-1} &= \mu\left(\frac{M_t^r}{P_t}\right)^{\rho-1} + \beta\left(C_{t+1}^r\right)^{\rho-1}\frac{1}{\pi_{t+1}} \\
(C_t^r)^{\rho-1} &= \mu\left(\frac{M_t^r}{P_t}\right)^{\rho-1} + \beta\left(C_t^r\right)^{\rho-1}\left[\beta\frac{R_t}{\pi_{t+1}}\right]^{-1}\frac{1}{\pi_{t+1}} \\
(C_t^r)^{\rho-1} &= \mu\left(\frac{M_t^r}{P_t}\right)^{\rho-1} + \left(C_t^r\right)^{\rho-1}\frac{1}{R_t} \\
(C_t^r)^{\rho-1}\left[1-\frac{1}{R_t}\right] &= \mu\left(\frac{M_t^r}{P_t}\right)^{\rho-1} \\
\mu\left(\frac{M_t^r}{P_t}\right)^{\rho-1} &= (C_t^r)^{\rho-1}\left[1-\frac{1}{R_t}\right] \\
\left(\frac{M_t^r}{P_t}\right)^{\rho-1} &= (C_t^r)^{\rho-1}\frac{1}{\mu}\left[1-\frac{1}{R_t}\right]
\end{aligned}$$

Finally, we obtain the following relationship between real money holdings and consumption:

$$\frac{M_t^r}{P_t^c} = C_t^r\left\{\frac{1}{\mu}\left[1-\frac{1}{R_t}\right]\right\}^{\frac{1}{\rho-1}}$$

$$\frac{M_t^r}{P_t^c} = C_t^r\left\{\mu^{-1}\left[\frac{R_t-1}{R_t}\right]\right\}^{\frac{1}{\rho-1}}$$

$$\frac{M_t^r}{P_t^c} = C_t^r\left\{\mu\frac{R_t}{R_t-1}\right\}^{\sigma}$$

Let me solve the model now. Let's define:

$$A_t^r \equiv K_t^r + \frac{B_t^r}{P_t^c} + \frac{P_t^F}{P_t^c}x_{F,t}^r + \frac{1}{R_t}\frac{M_t^r}{P_t}$$

I guess the following solution:

$$C_t^r + \frac{R_t-1}{R_t}\frac{M_t^r}{P_t} = \varepsilon_t\xi_t\left(\frac{R_{t-1}}{\pi_t}A_{t-1}^r\right)$$

Substituting this into the budget constraint, I obtain the evolution path to the aggregate retiree's assets

$$C_t^r + \frac{M_t^r}{P_t} + K_t^r + \frac{B_t^r}{P_t} + \frac{P_t^F}{P_t}x_{Ft}^r = \frac{1}{\gamma_t}\left(\frac{R_{t-1}}{\pi_t}A_{t-1}^r\right)$$

$$C_t^r + \left[\frac{R_t-1}{R_t} + \frac{1}{R_t}\right]\frac{M_t^r}{P_t} + K_t^r + \frac{B_t^r}{P_t} + \frac{P_t^F}{P_t}x_{Ft}^r = \frac{1}{\gamma_t}\left(\frac{R_{t-1}}{\pi_t}A_{t-1}^r\right)$$

$$C_t^r + \frac{R_t - 1}{R_t}\frac{M_t^r}{P_t} + \underbrace{\frac{1}{R_t}\frac{M_t^r}{P_t} + K_t^r + \frac{B_t^r}{P_t^c} + \frac{P_t^F}{P_t^c}x_{Ft}^r}_{\equiv A_t^r} = \frac{1}{\gamma_t}\left(\frac{R_{t-1}}{\pi_t}A_{t-1}^r\right)$$

$$\varepsilon_t\xi_t\left(\frac{R_{t-1}}{\pi_t}A_{t-1}^r\right) + A_t^r = \frac{1}{\gamma_t}\left(\frac{R_{t-1}}{\pi_t}A_{t-1}^r\right)$$

$$A_t^r = \left(\frac{R_{t-1}}{\pi_t}A_{t-1}^r\right) - \varepsilon_t\xi_t\left(\frac{R_{t-1}}{\pi_t}A_{t-1}^r\right)$$

$$A_t^r = (1 - \varepsilon_t\xi_t)\left(\frac{R_{t-1}}{\pi_t}A_{t-1}^r\right)$$

Now plugging this into the Euler equation

,

$$\frac{C_t^r}{C_{t+1}^r} = \left[\beta\frac{R_t}{\pi_{t+1}}\right]^{-\sigma}$$

Recall that

$$C_t^r + \frac{R_t - 1}{R_t}\frac{M_t^r}{P_t} = C_t^r\left[1 + \frac{R_t - 1}{R_t}\left\{\mu\frac{R_t}{R_t - 1}\right\}^{\sigma}\right] = \varepsilon_t\xi_t\left(\frac{R_{t-1}}{\pi_t}A_{t-1}^r\right)$$

And, therefore

$$C_t^r = \left[1 + \mu^{\sigma}\left\{\frac{R_t}{R_t - 1}\right\}^{\sigma-1}\right]^{-1}\varepsilon_t\xi_t\left(\frac{R_{t-1}}{\pi_t}A_{t-1}^r\right)$$

For the sake of notation, define:

$$\Psi_{t+1} \equiv \frac{\psi_t}{\psi_{t+1}} \equiv \frac{\left[1 + \mu^{\sigma}\left\{\frac{R_t}{R_t-1}\right\}^{\sigma-1}\right]^{-1}}{\left[1 + \mu^{\sigma}\left\{\frac{R_{t+1}}{R_{t+1}-1}\right\}^{\sigma-1}\right]^{-1}}$$

$$\frac{\varepsilon_t\xi_t\frac{1}{\gamma_t}\left(\frac{R_{t-1}}{\pi_t^c}A_{t-1}^r\right)}{\varepsilon_{t+1}\xi_{t+1}\frac{1}{\gamma_{t+1}}\left(\frac{R_t}{\pi_{t+1}^c}A_t^r\right)}\Psi_{t+1} = \left[\beta\frac{R_t}{\pi_{t+1}}\right]^{-\sigma}$$

$$\Psi_{t+1}\varepsilon_t\xi_t\frac{1}{\gamma_t}\left(\frac{R_{t-1}}{\pi_t}A_{t-1}^r\right) = \varepsilon_{t+1}\xi_{t+1}\frac{1}{\gamma_{t+1}}\left(\frac{R_t}{\pi_{t+1}}A_t^r\right)\left[\beta\frac{R_t}{\pi_{t+1}}\right]^{-\sigma}$$

$$\Psi_{t+1}\varepsilon_t\xi_t\frac{1}{\gamma_t}\left(\frac{R_{t-1}}{\pi_t}A_{t-1}^r\right) = \varepsilon_{t+1}\xi_{t+1}\left(\frac{1}{\gamma_{t+1}}\frac{R_t}{\pi_{t+1}}\left[(1 - \varepsilon_t\xi_t)\left(\frac{1}{\gamma_t}\frac{R_{t-1}}{\pi_t}A_{t-1}^r\right)\right]\right)\left[\beta\frac{R_t}{\pi_{t+1}}\right]^{-\sigma}$$

$$\Psi_{t+1}\varepsilon_t\xi_t\gamma_{t+1} = \varepsilon_{t+1}\xi_{t+1}\left(\frac{R_t}{\pi_{t+1}}\left[1 - \varepsilon_t\xi_t\right]\right)\left[\beta\frac{R_t}{\pi_{t+1}^c}\right]^{-\sigma}$$

$$1 - \Psi_{t+1}\frac{\varepsilon_t\xi_t}{\varepsilon_{t+1}\xi_{t+1}}\gamma_{t+1}\beta^{\sigma}\left(\frac{R_t}{\pi_{t+1}}\right)^{\sigma-1} = \varepsilon_t\xi_t$$

The retiree consumption solution is now done. Let's now solve his value function. I conjecture the value function solution is linearly dependent on consumption,

$$V_t^r = \Delta_t^r C_t^r$$

As it's a solution,

$$\begin{aligned}
V_t^r &= \left[(C_t^r)^\rho + \frac{\mu}{-} \left(\frac{M_t^r}{P_t} \right)^\rho + \beta\gamma_{t+1} \left(V_{t+1}^r \right)^\rho \right]^{\frac{1}{\rho}} \\
V_t^r &= \left[(C_t^r)^\rho + \mu \left(C_t^r \left\{ \mu \frac{R_t}{R_t - 1} \right\}^\sigma \right)^\rho + \beta\gamma_{t+1} \left(V_{t+1}^r \right)^\rho \right]^{\frac{1}{\rho}} \\
V_t^r &= \left[(C_t^r)^\rho \left[1 + \mu \left\{ \mu \frac{R_t}{R_t - 1} \right\}^{\sigma\rho} \right] + \beta\gamma_{t+1} \left(V_{t+1}^r \right)^\rho \right]^{\frac{1}{\rho}} \\
(\Delta_t^r)^\rho (C_t^r)^\rho &= (C_t^r)^\rho \left[1 + \mu^{1+\sigma\rho} \left\{ \frac{R_t}{R_t - 1} \right\}^{\sigma\rho} {}^{-\sigma\rho} \right] + \beta\gamma_{t+1} \left(\Delta_{t+1}^r C_{t+1}^r \right)
\end{aligned}$$

Using the Euler Equation,

$$(\Delta_t^r)^\rho (C_t^r)^\rho = (C_t^r)^\rho \left[1 + \mu^{1+\sigma\rho} \left\{ \frac{R_t}{R_t - 1} \right\}^{\sigma\rho} \right] + \beta\gamma_{t+1} \left(\Delta_{t+1}^r \right)^\rho \left[\left(\beta \frac{R_t}{\pi_{t+1}} \right)^\sigma \right]^\rho (C_t^r)^\rho$$

$$(\Delta_t^r)^\rho = \left[1 + \mu^{1+\sigma\rho} \left\{ \frac{R_t}{R_t - 1} \right\}^{\sigma\rho} \right] + \beta\gamma_{t+1} \left(\Delta_{t+1}^r \right)^\rho \left[\left(\beta \frac{R_t}{\pi_{t+1}} \right)^\sigma \right]^\rho$$

It's worth noticing that

$$\begin{aligned}
\sigma &= \frac{1}{1-\rho} \rightarrow \sigma\rho = \frac{\rho}{1-\rho} \\
\sigma\rho + 1 &= \frac{\rho}{1-\rho} + 1 = \frac{1}{1-\rho} = \sigma \\
\sigma - 1 &= \frac{\rho}{1-\rho} \rightarrow \sigma - 1 = \sigma\rho
\end{aligned}$$

Then,

$$(\Delta_t^r)^\rho = \left[1 + \mu^\sigma \left\{ \frac{R_t}{R_t - 1} \right\}^{\sigma-1} \right] + \beta^\sigma \left[\frac{R_t}{\pi_{t+1}} \right]^{\sigma-1} \gamma_{t+1} \left(\Delta_{t+1}^r \right)^\rho$$

It is straightforward to show that the solution to this difference equation is:

$$\Delta_t^r = \left(\frac{\varepsilon_t \xi_t}{\left[1 + \mu^\sigma \left\{ \frac{R_t}{R_t - 1} \right\}^{\sigma-1} \right]} \right)^{-\frac{1}{\rho}}$$

Because...

$$\frac{\left[1+\mu^{\sigma}\left\{\frac{R_t}{R_t-1}\right\}^{\sigma-1}\right]}{\varepsilon_t\xi_t}=\left[1+\mu^{\sigma}\left\{\frac{R_t}{R_t-1}\right\}^{\sigma-1}\right]+\beta^{\sigma}\left[\frac{R_t}{\pi_{t+1}}\right]^{\sigma-1}\gamma_{t+1}\frac{\left[1+\mu^{\sigma}\left\{\frac{R_{t+1}}{R_{t+1}-1}\right\}^{\sigma-1}\right]}{\psi_{t+1}\varepsilon_{t+1}\xi_{t+1}}$$

$$1=\varepsilon_t\xi_t+\beta^{\sigma}\left[\frac{R_t}{\pi_{t+1}}\right]^{\sigma-1}\gamma_{t+1}\frac{\varepsilon_t\xi_t}{\varepsilon_{t+1}\xi_{t+1}}\frac{\left[1+\mu^{\sigma}\left\{\frac{R_{t+1}}{R_{t+1}-1}\right\}^{\sigma-1}\right]}{\left[1+\mu^{\sigma}\left\{\frac{R_t}{R_t-1}\right\}^{\sigma-1}\right]}$$

$$1-\Psi_{t+1}\beta^{\sigma}\left[\frac{R_t}{\pi_{t+1}}\right]^{\sigma-1}\gamma_{t+1}\frac{\varepsilon_t\xi_t}{\varepsilon_{t+1}\xi_{t+1}}=\varepsilon_t\xi_t$$

Done.

7.2 The worker's problem

The worker's problem can be recursively written as:

$$V_t^w=\max\left\{(C_t^w)^{\rho}+\mu\left(\frac{M_t^w}{P_t}\right)^{\rho}+\beta\left[\omega_{t+1}V_{t+1}^w+(1-\omega_{t+1})\,V_{t+1}^r\right]^{\rho}\right\}^{\frac{1}{\rho}}$$

subject to

$$\begin{aligned}C_t^w &+\frac{M_t^w}{P_t}+K_t^w+\frac{B_t^w}{P_t}+\frac{P_t^F}{P_t}x_{Ft}^w=\\ =&\quad\frac{1}{\gamma_t}\left\{\left[R_t^K+(1-\delta)\right]K_{t-1}^w+\frac{R_{t-1}}{\pi_t}\frac{B_{t-1}^w}{P_{t-1}}+\frac{\left(P_t^F+D_t^F\right)}{P_t}x_{F,t-1}^w+\frac{1}{\pi_t}\frac{M_{t-1}^w}{P_{t-1}}\right\}\end{aligned}$$

First, let's define A_t^w:

$$A_t^w\equiv K_t^w+\frac{B_t^w}{P_t^c}+\frac{P_t^F}{P_t^c}x_{F,t}^w+\frac{1}{R_{t+1}}\frac{P_t^g}{P_t^c}M_t^w$$

The problem now becomes:

$$V_t^w=\max\left\{(C_t^w)^{\rho}+\mu\left(\frac{M_t^w}{P_t}\right)^{\rho}+\beta\left[\omega_{t+1}V_{t+1}^w+(1-\omega_{t+1})\,V_{t+1}^r\right]^{\rho}\right\}^{\frac{1}{\rho}}$$

subject to

$$C_t^w+\frac{R_{t+1}-1}{R_{t+1}}\frac{M_t^w}{P_t^c}+A_t^w=W_t+\frac{R_{t-1}}{\pi_t}A_{t-1}^r$$

7.3 Consumption Euler Equation

The FOC wrt A_t^w is:

$$(C_t^w)^{\rho-1} = \beta\left[\omega_{t+1}V_{t+1}^w + (1-\omega_{t+1})\,V_{t+1}^r\right]^{\rho-1}\left\{\omega_{t+1}\frac{\partial V_{t+1}^w}{\partial A_t^w} + (1-\omega_{t+1})\,\frac{\partial V_{t+1}^r}{\partial A_t^w}\right\}$$

The envelope theorem gives us:

$$\frac{\partial V_t^w}{\partial A_t^w} = (V_t^w)^{1-\rho}\,(C_t^w)^{\rho-1}\,\frac{R_{t-1}}{\pi_t}$$

and,

$$\frac{\partial V_t^r}{\partial A_t^w} = (V_t^r)^{1-\rho}\,(C_t^r)^{\rho-1}\,\frac{R_{t-1}}{\pi_t}$$

Substituting these into the FOC,

$$\begin{aligned}(C_t^w)^{\rho-1} &= \beta\left[\omega_{t+1}V_{t+1}^w + (1-\omega_{t+1})\,V_{t+1}^r\right]^{\rho-1}\times\\ &\times\left\{\omega_{t+1}\left(V_{t+1}^w\right)^{1-\rho}\left(C_{t+1}^w\right)^{\rho-1}\frac{R_t}{\pi_{t+1}} + (1-\omega_{t+1})\left(V_{t+1}^r\right)^{1-\rho}\left(C_{t+1}^r\right)^{\rho-1}\frac{R_t}{\pi_{t+1}}\right\}\end{aligned}$$

I use the previous conjecture, $V_t^r = \Delta_t^r C_t^r$, and also conjecture now that $V_t^w = \Delta_t^w C_t^w$.

$$\begin{aligned}(C_t^w)^{\rho-1} &= \beta\left[\omega_{t+1}V_{t+1}^w + (1-\omega_{t+1})\,V_{t+1}^r\right]^{\rho-1}\times\\ &\times\left\{\omega_{t+1}\left(\Delta_{t+1}^w C_{t+1}^w\right)^{1-\rho}\left(C_{t+1}^w\right)^{\rho-1}\frac{R_t}{\pi_{t+1}} + (1-\omega_{t+1})\left(\Delta_{t+1}^r C_{t+1}^r\right)^{1-\rho}\left(C_{t+1}^r\right)^{\rho-1}\frac{R_t}{\pi_{t+1}}\right\}\end{aligned}$$

$$\begin{aligned}(C_t^w)^{\rho-1} &= \beta\frac{R_t}{\pi_{t+1}}\left[\omega_{t+1}\Delta_{t+1}^w C_{t+1}^w + (1-\omega_{t+1})\,\Delta_{t+1}^r C_{t+1}^r\right]^{\rho-1}\times\\ &\times\left\{\omega_{t+1}\left(\Delta_{t+1}^w\right)^{1-\rho} + (1-\omega_{t+1})\left(\Delta_{t+1}^r\right)^{1-\rho}\right\}\end{aligned}$$

$$\begin{aligned}(C_t^w)^{\rho-1} &= \beta\frac{R_t}{\pi_{t+1}}\left[\omega_{t+1}C_{t+1}^w + (1-\omega_{t+1})\left(\frac{\Delta_{t+1}^r}{\Delta_{t+1}^w}\right)C_{t+1}^r\right]^{\rho-1}\times\\ &\underbrace{\times\left\{\omega_{t+1} + (1-\omega_{t+1})\left(\frac{\Delta_{t+1}^r}{\Delta_{t+1}^w}\right)^{1-\rho}\right\}}_{\equiv\Omega_{t+1}}\end{aligned}$$

$$(C_t^w)^{\rho-1} = \beta\Omega_{t+1}\frac{R_t}{\pi_{t+1}}\left[\omega_{t+1}C_{t+1}^w + (1-\omega_{t+1})\left(\frac{\Delta_{t+1}^r}{\Delta_{t+1}^w}\right)C_{t+1}^r\right]^{\rho-1}$$

$$C_t^w \left[\beta \Omega_{t+1} \frac{R_t}{\pi_{t+1}}\right]^{\frac{1}{1-\rho}} = \omega_{t+1} C_{t+1}^w + (1-\omega_{t+1}) \left(\frac{\Delta_{t+1}^r}{\Delta_{t+1}^w}\right) C_{t+1}^r$$

This is our Euler Equation.

7.4 Real money balances

Let's find out the relationship between real money balances and consumption for a worker. The FOC of interest is:

$$\begin{aligned}(C_t^w)^{\rho-1} &= \mu \left(\frac{M_t^w}{P_t}\right)^{\rho-1} + \beta \left[\omega_{t+1} V_{t+1}^w + (1-\omega_{t+1}) V_{t+1}^r\right]^{\rho-1} \times \\ &\times \left\{\omega_{t+1} \frac{\partial V_{t+1}^w}{\partial \left(\frac{P_t^g}{P_t^c} M_t^w\right)} + (1-\omega_{t+1}) \frac{\partial V_{t+1}^r}{\partial \left(\frac{P_t^g}{P_t^c} M_t^w\right)}\right\}\end{aligned}$$

The envelope theorem gives us:

$$\frac{\partial V_t^w}{\partial \left(\frac{M_{t-1}^w}{P_{t-1}^c}\right)} = (V_t^w)^{1-\rho} (C_t^w)^{\rho-1} \frac{1}{\pi_t}$$

and,

$$\frac{\partial V_t^r}{\partial \left(\frac{M_{t-1}^w}{P_{t-1}^c}\right)} = (V_t^r)^{1-\rho} (C_t^r)^{\rho-1} \frac{1}{\pi_t}$$

Putting these together we obtain:

$$\begin{aligned}(C_t^w)^{\rho-1} &= \mu \left(\frac{M_t^w}{P_t^c}\right)^{\rho-1} + \beta \frac{1}{\pi_{t+1}} \left[\omega_{t+1} V_{t+1}^w + (1-\omega_{t+1}) V_{t+1}^r\right]^{\rho-1} \times \\ &\times \left\{\omega_{t+1} \left(V_{t+1}^w\right)^{1-\rho} \left(C_{t+1}^w\right)^{\rho-1} + (1-\omega_{t+1}) \left(V_{t+1}^r\right)^{1-\rho} \left(C_{t+1}^r\right)^{\rho-1}\right\}\end{aligned}$$

I now conjecture that:

$$\begin{aligned}V_t^w &= \Delta_t^w C_t^w \quad \text{and} \quad V_t^r = \Delta_t^r C_t^r \\ \Delta_t^w &= \left(\frac{\xi_t}{\left[1+\mu^\sigma \left\{\frac{R_t}{R_t-1}\right\}^{\sigma-1}\right]}\right)^{-\frac{1}{\rho}} \rightarrow \frac{\Delta_t^r}{\Delta_t^w} = \frac{\left(\frac{\varepsilon_t \xi_t}{\left[1+\mu^\sigma \left\{\frac{R_t}{R_t-1}\right\}^{\sigma-1}\right]}\right)^{-\frac{1}{\rho}}}{\left(\frac{\xi_t}{\left[1+\mu^\sigma \left\{\frac{R_t}{R_t-1}\right\}^{\sigma-1}\right]}\right)^{-\frac{1}{\rho}}} = \varepsilon_t^{-\frac{1}{\rho}}\end{aligned}$$

Then,

$$\begin{aligned}(C_t^w)^{\rho-1} &= \mu\left(\frac{M_t^w}{P_t}\right)^{\rho-1} + \beta\frac{1}{\pi_{t+1}}\left[\omega_{t+1}\Delta_{t+1}^w C_{t+1}^w + (1-\omega_{t+1})\,\Delta_{t+1}^r C_{t+1}^r\right]^{\rho-1}\times\\ &\times\left\{\omega_{t+1}\left(\Delta_{t+1}^w C_{t+1}^w\right)^{1-\rho}\left(C_{t+1}^w\right)^{\rho-1} + (1-\omega_{t+1})\left(\Delta_{t+1}^r C_{t+1}^r\right)^{1-\rho}\left(C_{t+1}^r\right)^{\rho-1}\right\}\end{aligned}$$

$$\begin{aligned}(C_t^w)^{\rho-1} &= \mu\left(\frac{M_t^w}{P_t}\right)^{\rho-1} + \beta\frac{1}{\pi_{t+1}}\left[\omega_{t+1}\Delta_{t+1}^w C_{t+1}^w + (1-\omega_{t+1})\,\Delta_{t+1}^r C_{t+1}^r\right]^{\rho-1}\times\\ &\times\left\{\omega_{t+1}\left(\Delta_{t+1}^w\right)^{1-\rho} + (1-\omega_{t+1})\left(\Delta_{t+1}^r\right)^{1-\rho}\right\}\end{aligned}$$

$$\begin{aligned}(C_t^w)^{\rho-1} &= \mu\left(\frac{M_t^w}{P_t}\right)^{\rho-1} + \beta\frac{1}{\pi_{t+1}}\left(\Delta_{t+1}^w\right)^{\rho-1}\left[\omega_{t+1}C_{t+1}^w + (1-\omega_{t+1})\left(\frac{\Delta_{t+1}^r}{\Delta_{t+1}^w}\right)C_{t+1}^r\right]^{\rho-1}\times\\ &\times\left(\Delta_{t+1}^w\right)^{1-\rho}\left\{\omega_{t+1} + (1-\omega_{t+1})\left(\frac{\Delta_{t+1}^r}{\Delta_{t+1}^w}\right)^{1-\rho}\right\}\end{aligned}$$

$$\begin{aligned}(C_t^w)^{\rho-1} &= \mu\left(\frac{M_t^w}{P_t}\right)^{\rho-1} + \beta\frac{1}{\pi_{t+1}}\left[\omega_{t+1}C_{t+1}^w + (1-\omega_{t+1})\left(\frac{\Delta_{t+1}^r}{\Delta_{t+1}^w}\right)C_{t+1}^r\right]^{\rho-1}\times\\ &\underbrace{\times\left\{\omega_{t+1} + (1-\omega_{t+1})\left(\frac{\Delta_{t+1}^r}{\Delta_{t+1}^w}\right)^{1-\rho}\right\}}_{\equiv\Omega_{t+1}}\end{aligned}$$

Using the previous Euler Equation

$$\begin{aligned}(C_t^w)^{\rho-1} &= \mu\left(\frac{M_t^w}{P_t}\right)^{\rho-1} + \beta\frac{1}{\pi_{t+1}}\left[C_t^w\left[\beta\Omega_{t+1}\frac{R_t}{\pi_{t+1}}\right]^{\frac{1}{1-\rho}}\right]^{\rho-1}\times\\ &\underbrace{\times\left\{\omega_{t+1} + (1-\omega_{t+1})\left(\frac{\Delta_{t+1}^r}{\Delta_{t+1}^w}\right)^{1-\rho}\right\}}_{\equiv\Omega_{t+1}}\end{aligned}$$

$$(C_t^w)^{\rho-1} = \mu\left(\frac{M_t^w}{P_t}\right)^{\rho-1} + \beta\frac{1}{\pi_{t+1}}\Omega_{t+1}\left(C_t^w\right)^{\rho-1}\left[\beta\Omega_{t+1}\frac{R_t}{\pi_{t+1}}\right]^{-1}$$

$$(C_t^w)^{\rho-1} = \mu\left(\frac{M_t^w}{P_t}\right)^{\rho-1} + \left(C_t^w\right)^{\rho-1}\frac{1}{R_t}$$

$$(C_t^w)^{\rho-1}\left[1-\frac{1}{R_t}\right] = \mu\left(\frac{M_t^w}{P_t}\right)^{\rho-1}$$

Recall that $\frac{1}{\rho-1} = -\sigma$,

$$(C_t^w)\left[1-\frac{1}{R_t}\right]^{-\sigma} = \mu^{-\sigma}\left(\frac{M_t^w}{P_t}\right)$$

$$(C_t^w)\left[1-\frac{1}{R_t}\right]^{-\sigma} = \mu^{-\sigma}\left(\frac{M_t^w}{P_t}\right)$$

Finally,

$$\frac{M_t^w}{P_t} = C_t^w\left[\mu\frac{R_t}{1-R_t}\right]^{\sigma}$$

7.5 Solution

First, let's recall our A_t^w definition:

$$A_t^w \equiv K_t^w + \frac{B_t^w}{P_t^c} + \frac{P_t^F}{P_t^c}x_{F,t}^w + \frac{1}{R_t}\frac{M_t^w}{P_t}$$

I guess the following solution:

$$C_t^w + \frac{R_t-1}{R_t}\frac{M_t^w}{P_t} = \xi_t\left(\frac{R_{t-1}}{\pi_t^c}A_{t-1}^w + H_t^w\right)$$

And the decision rule for a retiree who just abbandoned the labor force is

$$C_t^r + \frac{R_{t+1}-1}{R_{t+1}}\frac{M_t^r}{P_t} = \varepsilon_t\xi_t\left(\frac{R_{t-1}}{\pi_t^c}A_{t-1}^w\right)$$

Where H_t^w is the net present value of non financial wealth, defined as:

$$H_t^w \equiv \sum_{\nu=0}^{\infty}\frac{\frac{W_{t+\nu}}{P_{t+\nu}}}{\Pi_{s=1}^{\nu}\left[\frac{\Omega_{t+s}R_{t+s-1}}{\omega_{t+s}\pi_{t+s}}\right]} = \frac{W_t}{P_t} + \frac{\omega_{t+1}\pi_{t+1}H_{t+1}^w}{\Omega_t R_t}$$

Let's combine our guess with our budget restriction:

$$C_t^w + \frac{R_{t+1}-1}{R_{t+1}}\frac{M_t^w}{P_t} + A_t^w = \frac{W_t}{P_t} + \frac{R_{t-1}}{\pi_t}A_{t-1}^r$$

$$\xi_t\left(\frac{R_{t-1}}{\pi_t^c}A_{t-1}^w + H_t^w\right) + A_t^w = \frac{W_t}{P_t} + \frac{R_{t-1}}{\pi_t}A_{t-1}^r$$

$$\xi_t\left(\frac{R_{t-1}}{\pi_t}A_{t-1}^w + \frac{W_t}{P_t} + \frac{\omega_{t+1}\pi_{t+1}H_{t+1}^w}{\Omega_t R_t}\right) + A_t^w = \frac{W_t}{P_t} + \frac{R_{t-1}}{\pi_t}A_{t-1}^r$$

$$\xi_t\left(\frac{\omega_{t+1}\pi_{t+1}H_{t+1}^w}{\Omega_t R_t}\right) + A_t^w = (1-\xi_t)\left[\frac{W_t}{P_t} + \frac{R_{t-1}}{\pi_t}A_{t-1}^r\right]$$

$$A_t^w = (1-\xi_t)\left[\frac{W_t}{P_t} + \frac{R_{t-1}}{\pi_t}A_{t-1}^r\right] - \xi_t\left(\frac{\omega_{t+1}\pi_{t+1}H_{t+1}^w}{\Omega_t R_t}\right)$$

$$A_t^w = (1-\xi_t)\left[\frac{W_t}{P_t} + \frac{R_{t-1}}{\pi_t}A_{t-1}^r\right] - \xi_t\left(\frac{\omega_{t+1}\pi_{t+1}H_{t+1}^w}{\Omega_t R_t}\right) + \left(\frac{\omega_{t+1}\pi_{t+1}H_{t+1}^w}{\Omega_t R_t}\right) - \left(\frac{\omega_{t+1}\pi_{t+1}H_{t+1}^w}{\Omega_t R_t}\right)$$

$$A_t^w + \frac{\omega_{t+1}\pi_{t+1}H_{t+1}^w}{\Omega_t R_t} = (1-\xi_t)\left[H_t^w + \frac{R_{t-1}}{\pi_t}A_{t-1}^r\right]$$

Which is how the workers' assets evolve. Now, getting back to our Euler equation,

$$C_t^w\left[\beta\Omega_{t+1}\frac{R_t}{\pi_{t+1}}\right]^{\frac{1}{1-\rho}} = \omega_{t+1}C_{t+1}^w + (1-\omega_{t+1})\left(\frac{\Delta_{t+1}^r}{\Delta_{t+1}^w}\right)C_{t+1}^r$$

Using our solution guess,

$$C_t^w\left[1 + \frac{R_{t+1}-1}{R_{t+1}}\left[\mu\frac{R_t}{1-R_t}\right]^\sigma\right] = \xi_t\left(\frac{R_{t-1}}{\pi_t}A_{t-1}^w + H_t^w\right)$$

$$\begin{aligned}
\xi_t &\quad \left(\frac{R_{t-1}}{\pi_t}A_{t-1}^w + H_t^w\right)\left[\beta\Omega_{t+1}\frac{R_t}{\pi_{t+1}}\right]^{\frac{1}{1-\rho}} \frac{1}{\left[1 + \frac{R_{t+1}-1}{R_{t+1}}\left[\mu\frac{R_t}{1-R_t}\right]^\sigma\right]} = \\
= &\quad \omega_{t+1}\xi_{t+1}\left(\frac{R_t}{\pi_t}A_t^w + H_{t+1}^w\right)\frac{1}{\left[1 + \frac{R_{t+2}-1}{R_{t+2}}\left[\mu\frac{R_{t+1}}{1-R_{t+1}}\right]^\sigma\right]} +
\end{aligned}$$

$$+ (1-\omega_{t+1})\left(\frac{\Delta_{t+1}^r}{\Delta_{t+1}^w}\right)\varepsilon_{t+1}\xi_{t+1}\left(\frac{R_t}{\pi_t}A_t^w\right)\frac{1}{\left[1 + \frac{R_{t+2}-1}{R_{t+2}}\left[\mu\frac{R_{t+1}}{1-R_{t+1}}\right]^\sigma\right]}$$

$$\begin{aligned}
\xi_t &\quad \left(\frac{R_{t-1}}{\pi_t}A_{t-1}^w + H_t^w\right)\left[\beta\Omega_{t+1}\frac{R_t}{\pi_{t+1}}\right]^{\frac{1}{1-\rho}} \underbrace{\frac{\left[1 + \frac{R_{t+2}-1}{R_{t+2}}\left[\mu\frac{R_{t+1}}{1-R_{t+1}}\right]^\sigma\right]}{\left[1 + \frac{R_{t+1}-1}{R_{t+1}}\left[\mu\frac{R_t}{1-R_t}\right]^\sigma\right]}}_{\equiv\Psi_{t+1}} = \\
= &\quad \omega_{t+1}\xi_{t+1}\left(\frac{R_t}{\pi_t}A_t^w + H_{t+1}^w\right) + (1-\omega_{t+1})\varepsilon_{t+1}\xi_{t+1}\left(\frac{\Delta_{t+1}^r}{\Delta_{t+1}^w}\right)\left(\frac{R_t}{\pi_t}A_t^w\right)
\end{aligned}$$

$$\begin{aligned}
\xi_t &\quad \left(\frac{R_{t-1}}{\pi_t}A_{t-1}^w + H_t^w\right)\left[\beta\Omega_{t+1}\frac{R_t}{\pi_{t+1}^c}\right]^\sigma \Psi_{t+1} = \\
= &\quad \omega_{t+1}\xi_{t+1}\left(\frac{R_t}{\pi_t^c}A_t^w + H_{t+1}^w\right) + (1-\omega_{t+1})\varepsilon_{t+1}\xi_{t+1}\left(\frac{\Delta_{t+1}^r}{\Delta_{t+1}^w}\right)\left(\frac{R_t}{\pi_t}A_t^w\right)
\end{aligned}$$

$$
\begin{aligned}
\xi_t \quad & \left(\frac{R_{t-1}}{\pi_t}A^w_{t-1} + H^w_t\right)\left[\beta\Omega_{t+1}\frac{R_t}{\pi_{t+1}}\right]^\sigma \Psi_{t+1} = \\
= \quad & \omega_{t+1}\xi_{t+1}\frac{R_t}{\pi_t}A^w_t + \omega_{t+1}\xi_{t+1}H^w_{t+1} + (1-\omega_{t+1})\,\xi_{t+1}\,(\varepsilon_{t+1})^{1-\frac{1}{\rho}}\left(\frac{R_t}{\pi_t}A^w_t\right)
\end{aligned}
$$

$$
\begin{aligned}
\tfrac{\xi_t}{\xi_{t+1}} \quad & \left(\frac{R_{t-1}}{\pi_t}A^w_{t-1} + H^w_t\right)\left[\beta\Omega_{t+1}\frac{R_t}{\pi_{t+1}}\right]^\sigma \Psi_{t+1} = \\
= \quad & \omega_{t+1}\frac{R_t}{\pi_t}A^w_t + (1-\omega_{t+1})\,(\varepsilon_{t+1})^{1-\frac{1}{\rho}}\left(\frac{R_t}{\pi_t}A^w_t\right) + \omega_{t+1}\xi_{t+1}H^w_{t+1}
\end{aligned}
$$

$$
\begin{aligned}
\tfrac{\xi_t}{\xi_{t+1}} \quad & \left(\frac{R_{t-1}}{\pi_t}A^w_{t-1} + H^w_t\right)\left[\beta\Omega_{t+1}\frac{R_t}{\pi_{t+1}}\right]^\sigma \Psi_{t+1} = \\
= \quad & \left[\omega_{t+1} + (1-\omega_{t+1})\,(\varepsilon_{t+1})^{1-\frac{1}{\rho}}\right]\frac{R_t}{\pi_t}A^w_t + \omega_{t+1}\xi_{t+1}H^w_{t+1}
\end{aligned}
$$

$$
\begin{aligned}
\tfrac{\xi_t}{\xi_{t+1}} \quad & \left(\frac{R_{t-1}}{\pi_t}A^w_{t-1} + H^w_t\right)\left[\beta\Omega_{t+1}\frac{R_t}{\pi_{t+1}}\right]^\sigma \Psi_{t+1} = \\
= \quad & \underbrace{\left[\omega_{t+1} + (1-\omega_{t+1})\,(\varepsilon_{t+1})^{\frac{\rho-1}{\rho}}\right]}_{\equiv\Omega_{t+1}}\frac{R_t}{\pi_t}A^w_t + \omega_{t+1}\xi_{t+1}H^w_{t+1}
\end{aligned}
$$

$$
\frac{\xi_t}{\xi_{t+1}}\left(\frac{R_{t-1}}{\pi^c_t}A^w_{t-1} + H^w_t\right)\left[\beta\Omega_{t+1}\frac{R_t}{\pi^c_{t+1}}\right]^\sigma \Psi_{t+1} = \Omega_{t+1}\frac{R_t}{\pi_t}A^w_t + \omega_{t+1}\xi_{t+1}H^w_{t+1}
$$

$$
\frac{\xi_t}{\xi_{t+1}}\left(\frac{R_{t-1}}{\pi_t}A^w_{t-1} + H^w_t\right)\beta^\sigma\left[\Omega_{t+1}\frac{R_t}{\pi_{t+1}}\right]^{\sigma-1} \Psi_{t+1} = A^w_t + \frac{\omega_{t+1}\xi_{t+1}H^w_{t+1}}{\Omega_{t+1}\frac{R_t}{\pi_t}}
$$

Let's recall the law of motion of workers's assets...

$$
A^w_t + \frac{\omega_{t+1}\pi_{t+1}H^w_{t+1}}{\Omega_t R_t} = (1-\xi_t)\left[H^w_t + \frac{R_{t-1}}{\pi_t}A^r_{t-1}\right]
$$

Putting these together,

$$
\frac{\xi_t}{\xi_{t+1}}\left(\frac{R_{t-1}}{\pi_t}A^w_{t-1} + H^w_t\right)\beta^\sigma\left[\Omega_{t+1}\frac{R_t}{\pi_{t+1}}\right]^{\sigma-1} \Psi_{t+1} = (1-\xi_t)\left[H^w_t + \frac{R_{t-1}}{\pi_t}A^r_{t-1}\right]
$$

$$
\frac{\xi_t}{\xi_{t+1}}\beta^\sigma\left[\Omega_{t+1}\frac{R_t}{\pi_{t+1}}\right]^{\sigma-1} \Psi_{t+1} = 1-\xi_t
$$

$$
\xi_t = 1 - \frac{\xi_t}{\xi_{t+1}}\beta^\sigma\left[\Omega_{t+1}\frac{R_t}{\pi_{t+1}}\right]^{\sigma-1} \Psi_{t+1}
$$

Done.

7.6
Aggregating workers and retirees

Let's recall our consumption solutions:

$$C_t^r \left[1 + \mu^\sigma \left\{\frac{R_t}{R_t - 1}\right\}^{\sigma-1}\right] = \varepsilon_t \xi_t \left(\frac{R_{t-1}}{\pi_t} A_{t-1}^r\right)$$

$$C_t^w \left[1 + \mu^\sigma \left\{\frac{R_t}{R_t - 1}\right\}^{\sigma-1}\right] = \xi_t \left(\frac{R_{t-1}}{\pi_t} A_{t-1}^w + H_t^w\right)$$

Let's also define λ_t as the share of total capital held by retirees. That is,

$$\lambda_t \equiv \frac{A_t^r}{A_t}$$

The aggregate consumption function is the sum of the solutions,

$$\begin{aligned}
C_t \left[1 + \mu^\sigma \left\{\frac{R_t}{R_t - 1}\right\}^{\sigma-1}\right] &= \varepsilon_t \xi_t \lambda_{t-1} \left(\frac{R_{t-1} A_{t-1}}{\gamma_t \pi_t}\right) + \xi_t \left(\frac{R_{t-1} A_{t-1}}{\pi_t}(1 - \lambda_{t-1}) + H_t^w\right) \\
C_t \left[1 + \mu^\sigma \left\{\frac{R_t}{R_t - 1}\right\}^{\sigma-1}\right] &= \xi_t \left\{\varepsilon_t \lambda_{t-1} \left(\frac{R_{t-1} A_{t-1}}{\gamma_t \pi_t}\right) + \frac{R_{t-1} A_{t-1}}{\pi_t}(1 - \lambda_{t-1}) + H_t\right\} \\
C_t \left[1 + \mu^\sigma \left\{\frac{R_t}{R_t - 1}\right\}^{\sigma-1}\right] &= \xi_t \left\{\lambda_{t-1} \frac{\varepsilon_t}{\gamma_t} \left(\frac{R_{t-1} A_{t-1}}{\pi_t}\right) - \lambda_{t-1} \frac{R_{t-1} A_{t-1}}{\pi_t} + \frac{R_{t-1} A_{t-1}}{\pi_t} + H_t\right\} \\
C_t \left[1 + \mu^\sigma \left\{\frac{R_t}{R_t - 1}\right\}^{\sigma-1}\right] &= \xi_t \left\{\lambda_{t-1} \left(\frac{R_{t-1} A_{t-1}}{\pi_t}\right) \left(\frac{\varepsilon_t}{\gamma_t} - 1\right) + \frac{R_{t-1} A_{t-1}}{\pi_t} + H_t\right\} \\
C_t \left[1 + \mu^\sigma \left\{\frac{R_t}{R_t - 1}\right\}^{\sigma-1}\right] &= \xi_t \left\{\frac{R_{t-1} A_{t-1}}{\pi_t} \left[\lambda_{t-1} (\varepsilon_t - 1) + 1\right] + H_t\right\}
\end{aligned}$$

Aggregate assets for retirees constitute of the remaining assets by those who were already retired and the savings that the newly retired have:

$$A_t^r = \frac{R_{t-1}}{\pi_t} A_{t-1}^r - C_t^r - \frac{R_t - 1}{R_t} \frac{M_t^r}{P_t} + (1 - \omega_{t+1}) \left(\frac{R_{t-1} A_{t-1}^w}{\pi_t} + \frac{W_t}{P_t} N_t^w - C_t^w - \frac{R_t - 1}{R_t} \frac{M_t^w}{P_t}\right)$$

$$A_t^r = \frac{R_{t-1}}{\pi_t} A_{t-1}^r - \varepsilon_t \xi_t \left(\frac{1}{\gamma_t} \frac{R_{t-1}}{\pi_t} A_{t-1}^r\right) + (1 - \omega_{t+1}) \left(\frac{R_{t-1} A_{t-1}^w}{\pi_t} + \frac{W_t}{P_t} N_t^w - \xi_t \left(\frac{R_{t-1}}{\pi_t^c} A_{t-1}^w + \frac{W_t}{P_t} N_t^w\right)\right)$$

$$A_t^r = \frac{R_{t-1}}{\pi_t} A_{t-1}^r - \varepsilon_t \xi_t \left(\frac{1}{\gamma_t} \frac{R_{t-1}}{\pi_t} A_{t-1}^r\right) + (1 - \omega_{t+1}) \left(\frac{R_{t-1} A_{t-1}^w}{\pi_t} + \frac{W_t}{P_t} N_t^w - \xi_t \left(\frac{R_{t-1}}{\pi_t^c} A_{t-1}^w + \frac{W_t}{P_t} N_t^w\right)\right)$$

As for workers,

$$A_t^w + \frac{\omega_{t+1} H_{t+1}^w}{\Omega_t R_t / \pi_{t+1}} = (1 - \xi_t) \left[H_t^w + \frac{R_{t-1}}{\pi_t} A_{t-1}^r\right]$$

$$A_t^w = \omega_{t+1} \left(\frac{R_{t-1} A_{t-1}^w}{\pi_t} + \frac{W_t N_t^w}{P_t} - C_t^w - \frac{R_t - 1}{R_t} \frac{M_t^w}{P_t}\right)$$

Putting the consumption solution for the retiree together with the last two equations, we obtain the law of motion of assets distribution:

$$C_t^r + \frac{R_t - 1}{R_t}\frac{M_t^r}{P_t} = \varepsilon_t \xi_t \left(\frac{1}{\gamma_t} \frac{R_{t-1}}{\pi_t} A_{t-1}^r \right)$$

$$\lambda_t A_t = \frac{R_{t-1}}{\pi_t} A_{t-1}^r - \varepsilon_t \xi_t \left(\frac{1}{\gamma_t} \frac{R_{t-1}}{\pi_t} A_{t-1}^r \right) + \frac{R_{t-1} A_{t-1}^w}{\pi_t} + \frac{W_t}{P_t} N_t^w - \xi_t \left(\frac{R_{t-1}}{\pi_t^c} A_{t-1}^w + \frac{W_t}{P_t} + \frac{\omega_{t+1}\pi_{t+1} H_{t+1}^w}{\Omega_t R_t} \right)$$

$$\lambda_t A_t = \left[1 - \frac{\varepsilon_t \xi_t}{\gamma_t} \right] \left(\frac{R_{t-1}}{\pi_t} \lambda_{t-1} A_{t-1} \right) + (1 - \omega_{t+1}) \left(\frac{(1-\lambda_t) A_t}{\omega_{t+1}} \right)$$

$$\lambda_t A_t \omega_{t+1} = \left[1 - \frac{\varepsilon_t \xi_t}{\gamma_t} \right] \omega_{t+1} \left(\frac{R_{t-1}}{\pi_t} \lambda_{t-1} A_{t-1} \right) + (1 - \omega_{t+1}) \left((1-\lambda_t) A_t \right)$$

$$\lambda_t A_t \omega_{t+1} = \left[1 - \frac{\varepsilon_t \xi_t}{\gamma_t} \right] \omega_{t+1} \left(\frac{R_{t-1}}{\pi_t} \lambda_{t-1} A_{t-1} \right) + (1 - \omega_{t+1}) A_t - \lambda_t A_t + \omega_{t+1} \lambda_t A_t$$

$$\lambda_t A_t = \left[1 - \frac{\varepsilon_t \xi_t}{\gamma_t} \right] \omega_{t+1} \left(\frac{R_{t-1}}{\pi_t} \lambda_{t-1} A_{t-1} \right) + (1 - \omega_{t+1}) A_t$$

$$\left[\lambda_t - (1 - \omega_{t+1}) \right] A_t = \omega_{t+1} \left[1 - \varepsilon_t \xi_t \right] \left(\frac{R_{t-1}}{\pi_t} \lambda_{t-1} A_{t-1} \right)$$

$$\lambda_t A_t \omega_{t+1} = \left[1 - \varepsilon_t \xi_t \right] \omega_{t+1} \left(\frac{R_{t-1}}{\pi_t} \lambda_{t-1} A_{t-1} \right) + (1 - \omega_{t+1}) A_t - \lambda_t (1 - \omega_{t+1}) A_t$$

$$\lambda_t A_t \omega_{t+1} = \left[1 - \varepsilon_t \xi_t \right] \omega_{t+1} \left(\frac{R_{t-1}}{\pi_t} \lambda_{t-1} A_{t-1} \right) + (1 - \omega_{t+1}) A_t - \lambda_t A_t + \lambda_t A_t \omega_{t+1}$$

$$\left[\lambda_t - (1 - \omega_{t+1}) \right] A_t = \lambda_{t-1} \omega_{t+1} \left[1 - \varepsilon_t \xi_t \right] \left(\frac{R_{t-1}}{\pi_t} A_{t-1} \right)$$

$$\left[\lambda_t - (1 - \omega_{t+1}) \right] A_t = \omega_{t+1} \left(1 - \xi_t^r \right) \lambda_{t-1} \frac{R_{t-1} A_{t-1}}{\pi_t}$$

As for the total money demand, let's write our cohort demands again:

$$\frac{M_t^r}{P_t} = C_t^r \left[\mu \frac{R_t}{1 - R_t} \right]^\sigma$$

$$\frac{M_t^w}{P_t} = C_t^w \left[\mu \frac{R_t}{1 - R_t} \right]^\sigma$$

It follows that the total money demand in the economy is the sum of these last two,

$$\frac{M_t^d}{P_t} = C_t \left[\mu \frac{R_t}{1 - R_t} \right]^\sigma$$

The monetary market equilibrium implies that:

$$M_t^d = M_t^s$$

7.7 Summary of model equations

Money demand

$$\frac{M_t^d}{P_t} = C_t \left[\mu \frac{R_t}{1-R_t}\right]^{\sigma}$$

Evolution of asset ratio

$$\left[\lambda_t - (1-\omega_{t+1})\right] A_t = \omega_{t+1}\left(1-\xi_t^r\right)\lambda_{t-1}\frac{R_{t-1}A_{t-1}}{\pi_t}$$

Consumption

$$C_t \left[1 + \mu^{\sigma}\left\{\frac{R_t}{R_t-1}\right\}^{\sigma-1}\right] = \xi_t \left\{\frac{R_{t-1}}{\pi_t}A_{t-1}\left[1+\lambda_{t-1}\left(\varepsilon_t - 1\right)\right] + H_t\right\}$$

Worker's marginal propensity to consume out of assets

$$\xi_t = 1 - \frac{\xi_t}{\xi_{t+1}}\beta^{\sigma}\left[\Omega_{t+1}\frac{R_t}{\pi_{t+1}}\right]^{\sigma-1}\Psi_{t+1}$$

Evolution of non-financial assets

$$H_t^w \equiv \frac{W_t}{P_t} + \frac{\omega_{t+1}H_{t+1}^w}{(1+n_{t+1})(1+x_{t+1})\Omega_t R_t/\pi_{t+1}}$$

Ratio

$$\Omega_{t+1} = \omega_{t+1} + \left(1-\omega_{t+1}\right)(\varepsilon)^{\frac{1}{1-\sigma}}$$

Retiree's marginal propensity to consume out of assets

$$1 - \Psi_{t+1}\frac{\varepsilon_t \xi_t}{\varepsilon_{t+1}\xi_{t+1}}\gamma_{t+1}\beta^{\sigma}\left(\frac{R_t}{\pi_{t+1}}\right)^{\sigma-1} = \varepsilon_t \xi_t$$

Ratio

$$\Psi_{t+1} = \frac{\left[1+\mu^{\sigma}\left\{\frac{R_{t+1}}{R_{t+1}-1}\right\}^{\sigma-1}\right]}{\left[1+\mu^{\sigma}\left\{\frac{R_t}{R_t-1}\right\}^{\sigma-1}\right]}$$

Evolution of capital

$$K_{t+1} = Y_t - C_t + (1-\delta)K_t$$

Aggregate assets

$$A_t = K_t + \frac{1}{R_t}\frac{M_t}{P_t} + \frac{P_t^F}{P_t}$$

7.8
Steady state

One can express the steady state conveniently as the following set of equations:

$$\Psi = 1$$

$$\xi = 1 - \beta^{\sigma}\left[\Omega\frac{R}{\pi}\right]^{\sigma-1}$$

$$1 - \gamma\beta^{\sigma}\left(\frac{R}{\pi}\right)^{\sigma-1} = \varepsilon\xi$$

$$\Omega = \omega + (1-\omega)\,\varepsilon^{\frac{1}{\sigma-1}}$$

$$\begin{aligned} R^k &= (1-\alpha)\frac{y}{k} + (1-\delta) \overset{y=k^{1-\alpha}}{=} (1-\alpha)\,k^{-\alpha} + (1-\delta) \\ k &= \left[\frac{1-\alpha}{R^k-(1-\delta)}\right]^{\frac{1}{\alpha}} \to y = \left[\frac{1-\alpha}{R^k-(1-\delta)}\right]^{\frac{1-\alpha}{\alpha}} \end{aligned}$$

$$\begin{aligned} \lambda - 1 + \omega &= \omega\,(1-\varepsilon\xi)\,\frac{R}{\pi}\frac{1}{(1+x)(1+n)}\lambda \\ \lambda\left[\omega\,(1-\varepsilon\xi)\,\frac{R}{\pi}\frac{1}{(1+x)(1+n)}\right] &= 1-\omega \\ \lambda &= \frac{(1-\omega)\,(1+x)\,(1+n)}{\omega\,(1-\varepsilon\xi)\,R/\pi} \end{aligned}$$

$$h = \frac{\alpha y}{1 - \frac{\omega(1+x)(1+n)}{\Omega R/\pi}}$$

$$c\left[1 + \mu^{\sigma}\left\{\frac{R}{R-1}\right\}^{\sigma-1}\right] = \varepsilon\xi\left\{\frac{R}{\pi}a\left[1+\lambda\,(\varepsilon-1)\right] + h\right\}$$

$$a = k + \frac{1}{R}c\left[\mu\frac{R}{R-1}\right]^{\sigma}$$

$$\psi = \frac{1-\omega}{1+n-\gamma}$$