Pontifícia Universidade Católica
do Rio de Janeiro

Cesar Augusto Mendonça Zambrano

Wealth inequality in heterogeneous agent models: the role of portfolio choice

Dissertação de Mestrado

Thesis presented to the Programa de Pós–graduação em Economia da PUC-Rio in partial fulfillment of the requirements for the degree of Mestre em Economia .

Advisor : Prof. Eduardo Zilberman
Co-advisor: Prof. Márcio Gomes Pinto Garcia

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Abstract

Mendonca Zambrano, Cesar Augusto; Zilberman, Eduardo (Advisor); Pinto Garcia, Márcio Gomes (Co-Advisor). **Wealth inequality in heterogeneous agent models: the role of portfolio choice**. Rio de Janeiro, 2019. 34p. Dissertação de mestrado – Departamento de Economia, Pontifícia Universidade Católica do Rio de Janeiro.

We introduce households' portfolio decisions in a heterogeneous agents model to evaluate how this affects wealth inequality. To do so, we alter the Krusell and Smith (1998) model, incorporating a decreasing returns to scale technology, so that the representative firm issues risk-free bonds to raise capital for production and distributes profits (or losses) to equity holders. We also make use of Epstein-Zin preferences to augment the model's equity premium, by increasing households risk aversion. The model is able to replicate stylized facts: (i) poorest households seldom participate in the equity markets; (ii) households allocate higher proportions of their savings to equity investments as they get wealthier; (iii) households' expected return on savings increases with wealth. Inequality of wealth does increase in the model with portfolio decisions. Nevertheless, the effect on wealth inequality is small due to the low level of equity premium generated by the model. The result in unchanged even when we set very high values for risk aversion, and it is related to the lack of consumption growth volatility delivered by this class of model. Finally, we document that taking into account endogenous portfolio decisions enhances the effects of other sources of inequality.

Keywords

Heterogeneous agents;   Wealth inequality;   Portfolio choice;

Resumo

Mendonca Zambrano, Cesar Augusto; Zilberman, Eduardo; Pinto Garcia, Márcio Gomes. **Desigualdade de riqueza em modelos com agentes heterogêneos: o papel da escolha de portfólio**. Rio de Janeiro, 2019. 34p. Dissertação de Mestrado – Departamento de Economia, Pontifícia Universidade Católica do Rio de Janeiro.

Introduzimos escolha de portfólio em um modelo com agentes heterogêneos para avaliar como isso afeta a desigualdade de riqueza. Para tanto, alteramos o modelo de Krusell e Smith (1998), incorporando uma tecnologia de produção com retornos decrescentes de escala, de forma que a firma representativa emite títulos de dívida para levantar capital para produção e depois distribui os lucros (ou prejuízos) para os acionistas. Também fazemos uso de preferências Epstein-Zin para aumentar o *equity premium* do modelo, aumentando a aversão ao risco dos agentes. O modelo é capaz de replicar fatos estilizados: (i) agentes mais pobres praticamente não participam do mercado de ações; (ii) os agentes investem proporções maiores de suas poupanças em ações, conforme ficam mais ricos; (iii) o retorno esperado da poupança dos agentes aumenta com a riqueza. A desigualdade de riqueza aumenta com a incorporação de escolha de portfólio dos agentes. No entanto, o impacto na desigualdade é pequeno devido ao baixo nível de *equity premium* gerado pelo modelo. Esse resultado se mantém mesmo quando estabelecemos valores muito altos para a aversão ao risco, e está relacionado à falta de volatilidade de consumo gerada por essa classe de modelos. Finalmente, documentamos que levar em conta decisões endógenas de portfólio potencializa os efeitos de outras fontes de desigualdade.

Palavras-chave

Agentes heterogêneos; Desigualdade de riqueza; Escolha de portfólio;

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1
Introduction

Aiyagari's (1994) heterogeneous agents model, on a baseline calibration, generates a wealth Gini of 0.38, well bellow the value observed in the U.S. economy, which figured around 0.83 in the past decades[1]. Several papers where dedicated to explore mechanisms that could reproduce the observed wealth inequality in a heterogeneous agents model, for example: Krusell and Smith (1998) hypothesized discount rate heterogeneity in an economy with aggregate uncertainty, Quadrini (2000) explored entrepreneurship opportunities, Castaneda et al. (2003) made use of life cycle elements and a wage process with high dispersion, De Nardi (2004) introduced a bequest motive for households savings. In the present work, we investigate whether incorporating households' portfolio decisions can generate sizable wealth inequality.

It is well established that wealthier households allocate a higher proportion of their savings to risky equity investments[2].Campanale (2007) surveyed the relation between households' wealth level and the share of savings they allocate to equity investments. He also estimated households' expected return on savings, per wealth level, according to this relation. We display his results in table 1.1. As the table shows, households in higher percentiles of the wealth distribution allocate a greater proportion of their savings to equity investments, enjoying a higher expected return on savings. This mechanism, which we will refer to as *the portfolio mechanism*, should increase wealth inequality.

Wealth Percentile	**Equity Share of Savings***	**Savings Return**
0-40	6.96%	0.90%
40-80	13.45%	1.81%
80-90	22.36%	2.60%
90-95	27.88%	3.10%
95-98	39.69%	4.02%
98-99	42.02%	4.14%
99-100	58.76%	5.56%

* Includes investments in publicly traded stocks as well as in private equity.

Table 1.1: Saving's profile per wealth level (adapted from Campanale 2007)

[1]Cowell et al. (2013).

[2]See for example: Carroll (2000), Campbell (2006), and Calvet and Sodini (2014).

To evaluate the impact of the portfolio mechanism on wealth inequality, we develop a model with portfolio decisions, by making two changes in the Krusell and Smith (1998) stochastic heterogeneous agents model: (i) introducing a decreasing return to scale technology and (ii) incorporating Epstein-Zin preferences. With a decreasing returns to scale and stochastic productivity, the representative firm issues risk-free bonds each period to raise capital for production, aiming to maximize it's expected profit for the next period. The realized profit (or loss) is distributed to the equity holders. Households endogenously allocate their savings between the risk-free bonds and the risky equity investments. This method allow us to create risk free assets in net supply for the households sector and to calibrate the equity to debt ratio in the economy. Then, we assess the impact of the portfolio mechanism on inequality, by comparing this model (the two-asset economy) with a similar economy where agents dispose of only one asset to save, like in most standard models.

The use of recursive preferences allows us to disentangle households' risk aversion from their elasticity of inter-temporal substitution (EIS). The inequality produced by the portfolio mechanism is directly related to the equity premium received by equity holders, which can be augmented by increasing households' risk aversion. Tallarini (2000) showed that separating risk aversion from the EIS is sufficient to generate a higher equity premium, through risk aversion, without generating excessively high interest rates in the economy. The same study by Tallarini shows, using U.S. data, that a risk aversion parameter of about 50 could explain the equity premium. Using such a high level of risk aversion is subject to criticism since several estimations[3] have shown that individual risk aversion is not much higher than five. Nonetheless, we make use of this method, since it's a straightforward way to increase the equity premium and does not create the undesirable effects associated with lowering the EIS. We note, beforehand, that setting high levels of risk aversion generates scenarios with low absolute levels of wealth inequality, which might seem strange. Nevertheless, our main analysis focus on the difference of wealth inequality between the economies with and without portfolio choice, not on their absolute level of inequality. We subsequently introduce heterogeneity of time preferences in the economic models, so that the model with portfolio choice generates the same level of wealth inequality that is observed in the data.

The two-asset model is able to qualitatively replicate the observed relation between portfolio choice, savings return and wealth level: (i) poorest

[3]See, for example, Chetty (2006), and Holt and Laury (2014) for a survey.

households seldom participate in the equity markets; (ii) households allocate higher proportions of their savings to equity investments as they get wealthier; (iii) Households' expected return on savings increases with wealth. The inequality of wealth do increase in the model with portfolio choice. Nevertheless, the impact on inequality is small due to the low level of equity premium generated by the two-asset model. This result is unchanged even when calibrating extreme values of risk aversion, and it is related to the lack of consumption growth volatility delivered by the model. We discuss how our results relates to Tallarini's findings on risk aversion and equity premium. The two-asset model with preference heterogeneity is able to replicate the wealth Gini of the U.S. economy, and it shows the existence of a positive interaction between the portfolio mechanism and additional sources of heterogeneity in generating inequality.

This paper is related to the literature on wealth inequality. De Nardi (2015) makes a comprehensive survey on this literature, where it is worth highlighting, in addition to the papers already mentioned: Hendricks (2005), that explores risk aversion heterogeneity; Ríos Rull et al. (2001) who study the impact of habit formation; Carroll (2000) who introduces wealth in the utility function of households; and Cagetti and De Nardi (2006) that mixes bequest motives and entrepreneurship in a heterogeneous agents model. Closest to our work is Campanale (2007), who assigned heterogeneous return rates to households' savings, as a function of their wealth level, and obtained an increment in wealth Gini of 0.09 in comparison to when the return rates on savings are homogeneous. Our contribution to this literature is to show that endogenous portfolio decision is a source of wealth inequality in a heterogeneous agents model, and that there is a positive cross effect between it and other sources of inequality.

Our work is also related to the literature that attempts to better reproduce the size of the equity premium in economic models. Cochrane (2017) reviews the main advances in this literature. It is also worth noting the work of Krusell and Smith (1997), that first introduced portfolio decisions in a stochastic heterogeneous agents model in order to study asset pricing in this setting. Additionally, Gomes and Michaelides (2008) make use of Epstein-Zin preferences, risk aversion heterogeneity and life cycle elements to generate an equity premium close to the values observed in the data. We add to this literature by showing that the high levels of risk aversion that Tallarini (2000) used to explain the equity premium, in a model with an exogenous stochastic process for consumption, are not able to deliver a high equity premium in a heterogeneous agents model, due to the lack of consumption growth volatility generated endogenously by this class of model.

The remainder of this paper is organized as follows: section two presents both economic models (with and without portfolio decisions) that we use to evaluate the portfolio mechanism; in section three we overview the computational procedure to approximate the solution of those models, section four describes the baseline calibration used to generate our results; section five presents the results for (i) a standard calibration of risk aversion, (ii) for a case in which risk aversion is set to a very high value, and (iii) interacting the portfolio mechanism with heterogeneity of preferences. In Section five we also discuss the lack of equity premium in the two-asset model and how the portfolio mechanism potentiate the effects of other sources of inequality. Section six concludes.

2
Models

As outlined, we access the impact of the portfolio mechanism on wealth inequality by comparing two model economies, one where households can save using riskless bonds and/or risky equity shares, and another where agents are only able to save using equity investments, like in most standard models. These economies share the same features, differing by the representative firm's problem and by the availability of assets households can use to save. We lay out both models in this section.

2.1
Economic Environment

The economic environment is based on Krusell and Smith (1998). Time is discrete and there is a unit continuum of infinitely lived households. Each period, every household receives a labor endowment, $\bar{l}$, and faces an idiosyncratic employment shock, $\epsilon \in \{0; 1\}$. In case $\epsilon = 1$, the household is employed and earns a salary w_t which is taxed at the rate τ_t, in case $\epsilon = 0$, it is unemployed and receives, as benefit, a proportion, μ, of current market wages: μw_t. The only role of government is to collect taxes from employed households and pay benefits to the unemployed ones, the amount of taxes paid by the employed workers have to cover for all the unemployment benefit, such that: $\tau_t = \mu u_t / L_t \bar{l}$, where u_t is unemployment rate and L_t is the total labor force at period t.

Departing from Krusell and Smith (1998), households exhibits Epstein-Zin preferences, represented by:

$$V_t = \left((1-\beta) c_t^{1-\rho} + \beta (E_t V_{t+1}^{1-\gamma})^{\frac{1-\rho}{1-\gamma}} \right)^{\frac{1}{1-\rho}}$$

The parameter γ controls the relative risk aversion while ρ is the inverse of the elasticity of inter-temporal substitution (EIS). As already mentioned, Tallarini (2000) shows that this separation is sufficient to make it possible to increase the model's equity premium (through altering risk aversion) without also increasing the model's risk free rate to unreasonable values, a relation that was regarded as the *risk-free rate puzzle*.

The representative firm possesses a Cobb-Douglas production technology with a scale factor η. Therefore, total output Y_t depends on productivity, Z_t, aggregate capital, K_t, and total labor, L_t, according to the function:

$$Y_t = Z_t(K_t^{\alpha} L_t^{1-\alpha})^{\eta}$$

Aggregate productivity, Z_t, is exogenous and stochastic, assuming two possible values: Z_l and Z_h. These values are related to a bad (low) and a good (high) state of the economy, respectively. The aggregate productivity and the idiosyncratic employment shock both follow a Markov process. The process is such that the probability of an agent i becoming unemployed or failing to find an employment ($e_t^i = 0$) is higher if the economy is in a bad state ($Z_t = Z_l$).

Every agent in the economy is a price taker and the output price is normalized to one.

2.2 The Two-Asset Economy

The scale factor is lower than unit ($\eta < 1$), so the representative firm has an expected profit. To raise capital for production, the firm issues risk-free bonds in the bonds market, which is remunerated at the market rate, $r_{t,t+1}$ (the notation stands for contracted at t and paid at $t+1$). The realized profit or loss is then absorbed by the firm's equity holders. There is a finite immutable unit mass amount of equity shares of the representative firm that is negotiated among households in the assets market.

The firm's problem: At period t, the firm chooses the amount of capital contracted for the next period (K_{t+1}), aiming to maximizes it's expected profit in $t+1$, solving:

$$\max_{K_{t+1}} E_t[Z_{t+1}(K_{t+1}^{\alpha} L_{t+1}^{1-\alpha})^{\eta} - w_{t+1}L_{t+1} - (r_{t,t+1} + \delta)K_{t+1}]$$

Then, at each period t, given it's contracted amount of capital, K_t, and the realized productivity, Z_t, the firm chooses the amount of labor employed, solving:

$$\max_{L_t} \quad Z_t(K_t^{\alpha} L_t^{1-\alpha})^{\eta} - w_t L_t - (r_{t-1,t} + \delta)K_t$$

After paying salaries, the promised interest rate, and recomposing the capital depreciation (δK_t), the company will have a profit or a loss to distribute to shareholders.

Households problem: At every period, each household i maximize it's recursive utility:

$$\begin{aligned}
V_t(b_t, s_t, e_t, Z_t, \lambda_t) &= \max_{\{c_t, b_{t+1}, s_{t+1}\}_t^{\infty}} \left((1-\beta)c_t^{1-\rho} + \beta(E_t V_{t+1}^{1-\gamma})^{\frac{1-\rho}{1-\gamma}} \right)^{\frac{1}{1-\rho}} \\
& s.a. \\
c_t + b_{t+1} + q_t s_{t+1} &\leq (1 + r_{t-1,t})b_t + w_t(1-\tau)e_t\bar{l} + \mu w_t(1-e_t) + \\
& + (q_t + \pi_t)s_t \\
b_{t+1}, s_{t+1} &\geq 0 \\
\lambda_{t+1} &= G(Z_t, \lambda_t, Z_{t+1})
\end{aligned}$$

Where b_t and s_t are, respectively, the amount of bonds and equity shares that household i has at the beginning of the period (we drop i from the notation). Each unit of bond earns the interest rate $r_{t-1,t}$, determined previously, and each unit of equity share, s_t, have a market value of q_t. Also, for each unity of equity share, households receive a participation, π_t on the profit (or loss) of the representative firm. At each period, Household i chooses the consumption for the period, c_t, and the amount of bonds and equity shares for the next period: b_{t+1} and s_{t+1}. Agents can't borrow nor sell short equity shares.

Prices are determined by the interaction between agents in the economy, in such a way that, in order to compute and predict prices, every agent must keep track of the λ_t distribution of households over the state-space (ϵ_t,b_t,s_t). The evolution of the λ distribution depends on the transition between aggregate states Z, occurring according to the endogenous function: $\lambda_{t+1} = G(Z_t, \lambda_t, Z_{t+1})$.

Note that we can rewrite the first restriction in the households problem as:

$$c_t + b_{t+1} + q_t s_{t+1} \leq W_t + w_t(1-\tau)e_t\bar{l} + \mu w_t(1-e_t)$$
$$W_t = (1 + r_{t-1,t})b_t + (q_t + \pi_t)s_t$$

In this way, the relevant state for the household's problem is (Z_t,λ_t,W_t,e_t), where W_t is the total financial wealth that the individual household has at the begging of the period. It is not relevant, for the household decision, if that wealth came from any particular mix of bonds plus interests or equity shares plus dividends. This change of variable greatly reduces the complexity of the problem when it comes to approximating the solution computationally.

Recursive competitive equilibrium, two-asset economy. Denote households decision functions on the amount of bonds and equity shares to be carried to the next period as f^b and f^s, respectively. A recursive competitive equilibrium is composed by a law of motion G, the household value and policy functions, V,f^b and f^s and the pricing functions r, w and q such that $(V,\ f^b,\ f^s)$ solves the household's problem; w and r are competitive (which means they are given by the marginal productivity or expected marginal productivity of the representative firm); G is generated by f^b and f^b; the aggregate capital in each period equals total bonds demand of the previous period:

$$\int f^b(Z_t, \lambda_t, W_t, e_t)d\lambda_t = K_{t+1}$$

and the equity markets clears:

$$\int f^s(Z_t, \lambda_t, W_t, e_t)d\lambda_t = 0$$

2.3 The Single-Asset Economy

In this economy, the scale factor equals unit ($\eta = 1$) and the representative firm operates just like in a standard model, employing households' savings as equity capital for production, which is then remunerated at the rate: $r_t^k = \alpha Z_t(K_t/L_t)^{\alpha-1}$, for a given amount of K_t, L_t, and a realization of Z_t. Households choose their saving's level based on the expected return of equity.

Households problem: At every period, each household i solves:

$$
\begin{aligned}
V_t(k_t, e_t, Z_t, \Gamma_t) = \max_{\{c_t, k_{t+1}\}_t^\infty} &\left((1-\beta)c_t^{1-\rho} + \beta(E_t V_{t+1}^{1-\gamma})^{\frac{1-\rho}{1-\gamma}}\right)^{\frac{1}{1-\rho}} \\
s.a. & \\
c_t + k_{t+1} + \leq (1+r_t^k)k_t &+ w_t(1-\tau)e_t\bar{l} + \mu w_t(1-e_t) \\
k_{t+1} &\geq 0 \\
\Gamma_{t+1} &= H(Z_t, \Gamma_t, Z_{t+1})
\end{aligned}
$$

Now, the only decision to be made by households is the allocation of resources between consumption, c_t, and equity investments, k_t. They can't sell equities short. In order to compute price and form expectations, agents must keep track of households distribution Γ_t in the state space (ϵ_t,k_t), which evolves according to the endogenous law of motion: $\Gamma_{t+1} = H(Z_t, \Gamma_t, Z_{t+1})$.

Recursive competitive equilibrium, single asset economy. Denote households decision functions on the amount of equities to be carried to the next period as f^k. A recursive competitive equilibrium is composed by a law of motion H, the household value and policy functions, V and f^k, and the pricing functions r^k, w, such that (V, f^k) solves the household's problem; w and r^k are competitive (which means they are given by the marginal productivity or expected marginal productivity of the representative firm); H is generated by f^k; and the aggregate capital in each period equals total equity demand of the previous period: $\int f^k(Z_t, \Gamma_t, W_t, e_t)d\Gamma_t = K_{t+1}$.

3
Computation Strategy

Solving the two assets model with λ as a state variable in agents' problems is not possible because λ is an infinite dimensional object. Instead, we approximate the solution using Krusell and Smith (1997) algorithm. The idea is to use a finite set of moments of λ, $\boldsymbol{m}^{\boldsymbol{\lambda}}$, instead of the whole distribution as a state variable. In order to do so, we replace λ by two functions that dictate how agents perceive the economy, one to predict aggregate capital, K_{t+1}, and one pricing function for the price of shares, q_t:

$$K_{t+1} = H_1(\boldsymbol{m}^{\boldsymbol{\lambda_t}}, Z_t)$$
$$q_t = H_2(\boldsymbol{m}^{\boldsymbol{\lambda_t}}, Z_t)$$

The two functions are sufficient for agents to compute prices and form expectations. A good approximation is obtained when the predictions made by the two perceived functions are close enough to the values actually observed in the economy (obtained by simulation). The algorithm used to find the perceived functions, for the two assets model, works as follows:

1. Propose an initial guess for the functional forms of the two functions that dictates agents' perception.

2. Approximate households' value and policy functions using the perceived functions to orient agents' decisions.

3. Simulate an economy with a large number of households for T periods. Compute the aggregate capital sequence, $\{K\}^T$, and share prices, $\{q\}^T$, realized during the T simulated periods.

4. Discard the data of the first d periods and use the remaining data to update the parameters of the law of motion and the pricing function.

5. Go back to (2) and repeat the process with the updated perceived functions. Keep iterating in this manner until the perceived functions converge by some metric.

6. After the convergence of the perceived functions, check if the values predicted by them are close enough to the values actually obtained, by simulation, in the economy. If they are, stop. If not, go back to (1) and propose new functional forms for the perceived functions (usually adding other moments of λ into the specification).

For this model, as in Krusell and Smith (1997), using only the aggregate capital to represent λ, in both functions, produces a good result. Proposing a log-log functional forms in (1) also provided good results. In step (4), we use an OLS regression on the sequences of aggregate capital and equity share prices to update the parameters of the functional forms.

When approximating the households solution, in step (2), we perform a value function iteration, employing a two steps procedure: First, for a given total amount a household decides to save, we find the optimal mix between bonds and equity shares that she would choose to invest. Then, in a second step, we solve for household's wealth allocation between consumption and total savings, given how the total savings will be invested. In the second step we are able to employ the endogenous grid point method, introduced by Carroll (2006), which speeds up computation considerably.

In (3), we simulate the economy using 10.000 households, and a time length, T, of 1.650 quarters, discarding the first 150. Computing the aggregate capital, k_t, of each period is trivial (just integrate total households bond holdings of the previous period), but to find the realized q_t, we need to iterate the price of equity shares until market clearing condition holds. Promoting market clearing in every period of the simulation is computationally demanding.

To approximate the solution of the model with one asset, we employ the exact same algorithm as Krusell and Smith (1998), which is a simpler version of the one that was just described. In this case, there is only one perceived function for aggregate capital:

$$K_{t+1} = Q(\boldsymbol{m}^{\boldsymbol{\lambda}_t}, Z_t)$$

Approximating households value and policy functions is far simpler in this case because there is no portfolio choice in their problem. Also, the procedure is a lot faster because there is no need to search for the price of equity shares that generates market clearing in each period of the simulation. We omit the details about the procedures in this simpler case.

4
Baseline Calibration

Unless otherwise stated, we calibrate the models following Den Haan et al. (2010), except for the parameters of the recursive preference and the scale factor. Table 4.1 displays a list of parameters with their assigned values, which are standard for a quarterly economy.

Parameter		**value**	**comment**
Discount rate	β	0.99	Quarterly economy
Output elasticity of capital	α	0.36	Standard
Depreciation rate	δ	0.025	Quarterly economy
Time endowment	$\bar{l}$	1/0.9	Aggregate labor 1 in a bad state
Unemployment benefit	μ	0.15	

Table 4.1: Parameters values for a quarterly economic model

The individual employment shock and the aggregate productivity shock follow a joint Markov process with four possible outcomes from the perspective of individual households. The transition probabilities of the Markov chain are calibrated in such a way that:

- The persistence of a good or bad state of the economy are both equal to eight quarters.
- An unemployment spell lasts, on average, 1.5 quarters in a good state of the economy, and 2.5 quarters in a bad state of the economy.
- The unemployment rate, u, is 4% in a good state of the economy, and 10% in a bad state of the economy, no matter which was the previous state of the economy.

Denote by $p_{\epsilon\epsilon',ZZ'}$, the probability that a household will go from employment situation ϵ to ϵ' and the economy will go from aggregate state Z to Z', with $\epsilon, \epsilon' \in \{0,1\}$ and $Z, Z' \in \{Z_l, Z_h\}$. Then, following the aforementioned criteria, the following transition matrix is obtained for the joint Markov process:

$$P = \begin{bmatrix} p_{00,Z_lZ_l} & p_{01,Z_lZ_l} & p_{00,Z_lZ_h} & p_{01,Z_lZ_h} \\ p_{10,Z_lZ_l} & p_{11,Z_lZ_l} & p_{10,Z_lZ_h} & p_{11,Z_lZ_h} \\ p_{00,Z_hZ_l} & p_{01,Z_hZ_l} & p_{00,Z_hZ_h} & p_{01,Z_hZ_h} \\ p_{10,Z_hZ_l} & p_{11,Z_hZ_l} & p_{10,Z_hZ_h} & p_{11,Z_hZ_h} \end{bmatrix} = \begin{bmatrix} 52.5000 & 35.0000 & 03.1250 & 09.3750 \\ 03.8889 & 83.6111 & 0.02083 & 12.2917 \\ 09.3750 & 03.1250 & 29.1667 & 58.3333 \\ 0.09115 & 11.5885 & 02.4306 & 85.0694 \end{bmatrix} \%$$

In the two-asset model, we have to calibrate the scale factor η. In order to do it, we can target the following ratios in the model: the standard relative deviation of corporate profits and the equity/debt market size relation. Table 4.2 shows some possible values for η and the resulting values for those ratios, approximately. Note also that the ratio of expected profits per total sales, will be given by $1 - \eta$, by construction.

η	**RSD of Profit**	**Equity/Debt**
0.99	0.75	10%
0.98	0.38	20%
0.97	0.26	30%
0.96	0.19	40%
0.95	0.15	50%

Table 4.2: Scale factor values and economic ratios

Using data from the FRED, we estimated the relative standard deviation (RSD) of U.S. corporate sector's profit to be 0.13 from 1980 to 2018. Data from the FED shows that the U.S. ratio of publicly traded stocks value (equity) per total bonds value (total debt) ranged from 50% to 75% between 2011-2017. Therefore, we use an η value of 0.95, which results in a good approximation of both ratios.

Lower levels of η would, actually, attain a better fit to the average size of the equity market. However, as we calibrate lower levels of η, the number of iterations necessary to achieve convergence of the perceived functions, during the computational procedures, increases considerably. This issue is related to the size of the equity market: as it becomes bigger, each update in the pricing function of equities causes a greater impact on the other variables of the economy - and particularly on the level of aggregate capital - so more iterations are necessary to obtain a solution.

In the one-asset model, we calibrate the aggregate productivity for each state of the economy as $Z_l = 0.99$ and $Z_h = 1.01$, also following Den Haan et al. (2010). In the two-asset model, since the scale factor η reduces total output, we re-scale the aggregate productivity in such a way that, for the mean level of capital around which the economy fluctuates, output remains equal in both models. The re-scaled productivity, for $\eta = 0.95$, are: $Z_l = 1.05076$ and $Z_h = 1.07198$.

5
Results

5.1
Standard Calibration

First, we solve both models with a risk aversion of 5 and an elasticity of inter-temporal substitution of 0.5. Yogo (2004) estimated, for eleven countries, values for the EIS in the range 0-0.5. We use the high bound of this range, which is close to the values usually calibrated in macroeconomic models. We focus the discussion on the results of the two-asset model.

Employing the algorithm described in section 3 for the two-asset economy, we obtain the following perceived functions:

$$\log(K_{t+1}) = 0.1030 + 0.9696\log(K_t), \quad if \quad Z = Z_l, \quad R^2 = 0.9848$$
$$\log(K_{t+1}) = 0.1186 + 0.9668\log(K_t), \quad if \quad Z = Z_h, \quad R^2 = 0.9798$$

$$\log(q_t) = 0.6887 + 0.6091\log(K_t), \quad if \quad Z = Z_l, \quad R^2 = 0.9999$$
$$\log(q_t) = 0.7281 + 0.6006\log(K_t), \quad if \quad Z = Z_h, \quad R^2 = 0.9999$$

The first two equations describe agents' (households and representative firm) prediction for the next period's aggregate capital level, based on the aggregate capital level and state of the economy they observe in the current period. The second pair of equations represents the function that describes the price that agents attribute to equity shares, also based on the current aggregate capital level and state of the economy. These equations are enough for agents to solve their problems.

The R-squared is a measure of how well those perceived functions describe the true behavior of the economy. It is the R-squared obtained, during the computational procedures, in the last OLS regression performed to update the perceived functions (step 4 in the algorithm), before the convergence criteria is attained. As we can see, the R-squared indicates a close to perfect fit.

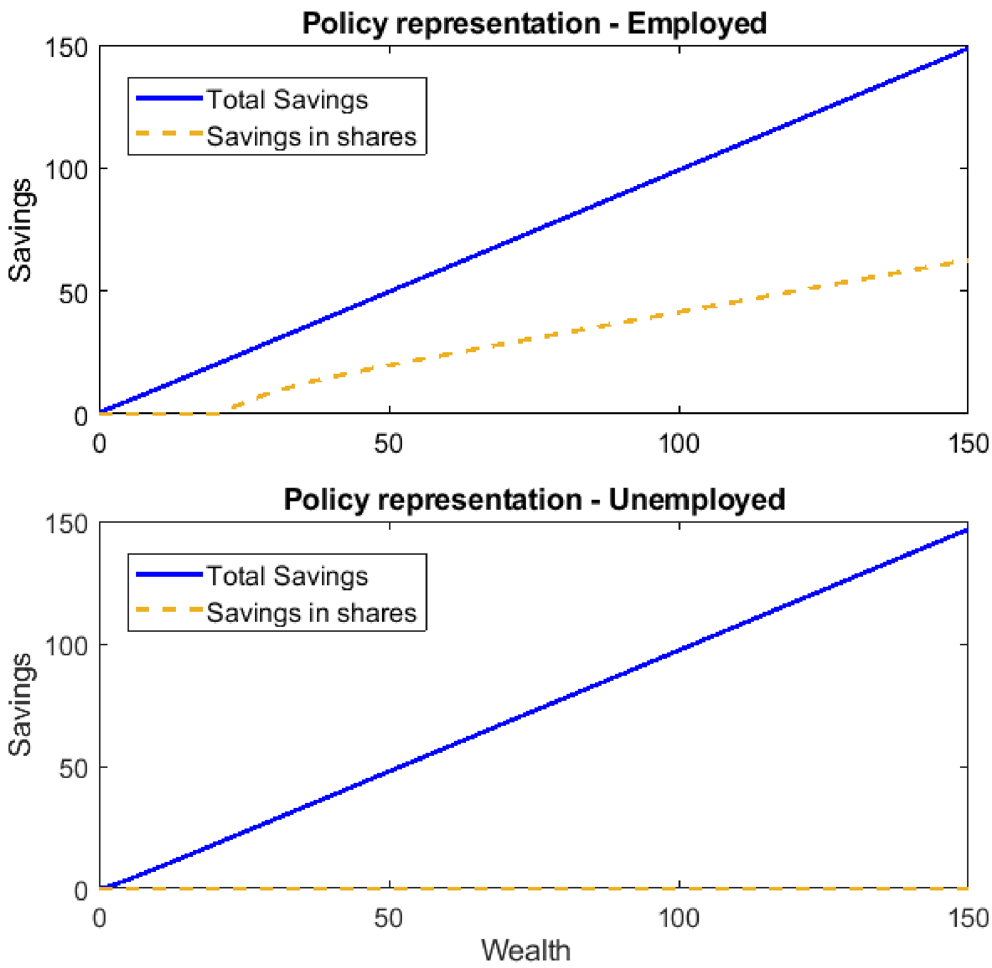
Figure 5.1: Share prices (q_t), per aggregate capital level and aggregate state

As these equations show, when the economy's aggregate state changes from a bad (good) state to good (bad) one, the aggregate capital for the next period and the price of equity shares both 'jump' to a higher (lower) value. Figure 5.1 illustrates the behavior of equity share prices, for both possible states of the economy and for various levels of aggregate capital.

Figure 5.2 shows a representation of households policy function, for a good state of the economy and a given level of aggregate capital. The straight line represent households total savings and the dashed line represents the amount of savings households allocate to equity investments. Therefore, the amount of savings allocated to bonds is the difference between the straight and dashed lines.

As illustrated, households with a low level of wealth do not invest in equity shares. Employed households start investing in equity shares after attaining a wealth level of 20.6. Then, the ratio of equity investments to total savings increases with wealth, until a point after which this ratio becomes constant. Qualitatively, the employed households policy function replicates the documented empirical relation between portfolio choice and wealth level. This result is obtained only due to the differences in the curvature of households value function, across different levels of wealth. At lower levels, households are close to the borrowing constraint, where they capability to sustain their consumption, in the face of a bad idiosyncratic shock, becomes impaired, so they choose to save in safe assets. At higher levels of wealth, far from the borrowing constraint, households portfolio decisions reflect the constant relative risk aversion of the Epstein-Zin preference, so they allocate a constant proportion of their savings to risky equity investments.

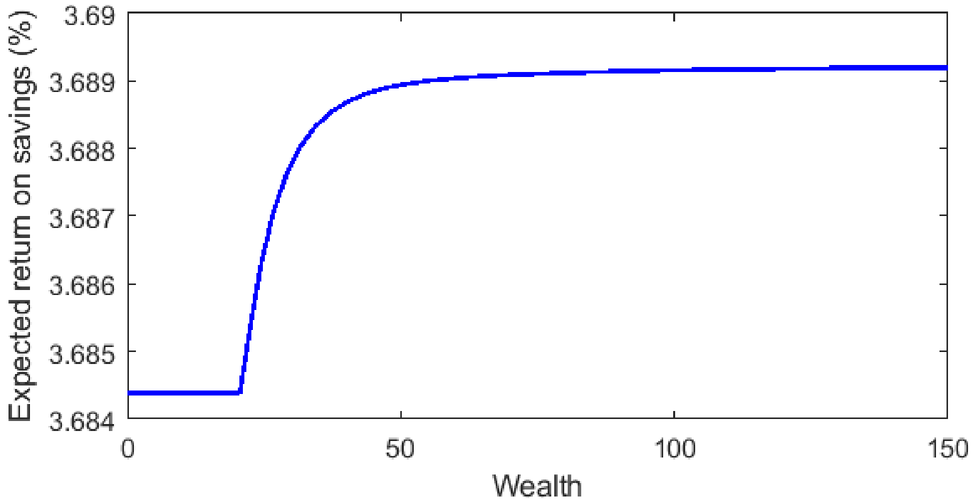
Figure 5.2: Household's policy representation, in a good aggregate state



Unemployed households have a stronger preference for safe assets. Only with a wealth level higher than about 221, which was not attained by any agent in our simulations, an unemployed household would start demanding equity investments. This feature is related to the persistence of the unemployment spell and creates a scenario where unemployed households hold no equity whatsoever, which is at odds with empiric observation. Despite this shortfall, the two-asset model successfully incorporates the portfolio mechanism, as households exposure to equity investments is positively correlated with their wealth level, and a higher exposure to equities results in a savings portfolio with a greater expected return.

Figure 5.3 illustrates the relation between expected return on savings and wealth level, for employed households. At low levels of wealth, expected return is constant and equal to the risk-free rate, because households invest in bonds only. As households become wealthier, their expected return on savings increases until a certain point after which it stays constant. This relation merely

Figure 5.3: Employed household's expected return, by wealth level



reflects the mix of bonds and equity investments those agents choose to allocate their savings to, according to their wealth level, as illustrated by the policy function representation.

The relation between expected return on savings and wealth level is a distinct feature of the model with two assets. We assess the impact of this relation on wealth inequality by comparing the wealth Gini of this model with the wealth Gini of the one-asset model, after submitting both models to the same sequences of aggregate and idiosyncratic shocks.

Table 5.1 shows the comparison between the two model economies. The values shown in the table are the average of the last 200 periods (one generation of 50 years) of the simulations. As we can see, wealth inequality is indeed higher in the two-asset model, but the difference in inequality is small. Wealth Gini is 0.2411 in the model with portfolio choice against 0.2278 in the model without it. The lack of substantial impact of the portfolio mechanism on wealth inequality is due to the very small equity premium delivered by the two-asset model, which is only 0.012%, approximately.

	Two-Asset Model	**One-Asset Model**
Wealth Gini	0.2411	0.2278
Equity/Debt ratio	51,2%	-
Equity premium (p.a.)	0.012%	-
Risk-free rate (p.a.)	4.032%	-

Table 5.1: Comparison between model economies

The lack of equity premium in standard macro models is known as the *equity premium puzzle*. A branch of the literature on macroeconomics is dedicate to reproduce the equity premium in economic models. To better gauge the impact of the portfolio mechanism on the real economy, we take the most straightforward approach to increase the equity premium, adjusting the parameter of risk aversion.

5.2 Pushing Risk Aversion

In this section we present the results obtained when we solve both models with a risk aversion parameter, γ, of 25. It is only half of the value that Tallarini (2000) used to conciliate the equity premium with the data on U.S. consumption growth, but it is sufficient for us to evaluate the effects, on our results, of pushing risk aversion to high levels. Setting even higher values of risk aversion in the two-asset model is very challenging[1].

Table 5.2 shows the results for this calibration. The most striking feature of both economies is the very low absolute level of wealth inequality. Due to the high level of risk aversion, all households increase their savings, but poorest agents increase their savings proportionally more, because their are closest to the borrowing constraint. As result, the absolute level of wealth inequality decreases. The main objective of this exercise, nonetheless, is to evaluate the difference in inequality between the economies with and without portfolio choice, when we increase the equity premium, rather than focusing on the absolute level of inequality. Furthermore, the level of wealth inequality can be adjusted in a straightforward manner, to match the empiric observation, by introducing discount rate heterogeneity.

	Two-Asset Model	**One-Asset Model**
Wealth Gini	0.0926	0.0707
Equity/Debt ratio	53.8%	-
Equity premium (p.a.)	0.050%	-
Risk-free rate (p.a.)	3.741%	-

Table 5.2: Comparison between model economies, $\gamma = 25$

The two-asset model, with a risk aversion parameter of 25 still does not generates an equity premium comparable to the 6% per year measured in the data. Hence, the inequality generated by the portfolio mechanism remains small. The difference in wealth inequality between the two models, using this

[1]To approximate households' value function, we use cubic spline interpolation and Newton-Raphson procedures to solve the maximization problem in each iteration. This method allows for a fast convergence of the value function but it struggles to map the point of maximum when the curvature of the utility function reflects a very high level of risk aversion.

calibration, is 0.0219, not much higher than the difference of 0.0133 obtained with the calibration of the previous section. Why the equity premium is so low in the two-asset model, even though we use high levels of risk aversion?

5.3 Equity Premium and Consumption Growth Volatility

Hansen and Jagannathan (1991) investigated the properties that a stochastic discount factor should have to conciliate the return on stock markets with the risk-free rate of return in the U.S. economy, over the period 1948-2005. They found bounds on the first and second moments of the stochastic discount factor, that would be consistent with the data on assets returns. Tallarini (2000) showed that, with Epstein-Zin preferences, those bounds are attainable with a value of risk aversion as high as 50. To do so, he departed from an Euler equation, and proposed a stochastic process for the quarterly consumption growth, which he calibrated with the standard deviation of consumption growth measured in the U.S. economy, from 1948 to 2005. He, then, arrived at the values of risk aversion that could reconcile the data on consumption with the Hansen-Jagannathan bounds.

The standard deviation of consumption growth in the U.S. economy, measured by Tallarini, was 5.27×10^{-1}. In our two-asset model, the standard deviation of consumption growth is about 2.74×10^{-3}, more than two orders of magnitude lower than the volatility of consumption in the real economy. Such a low level of consumption volatility implies a low market price of risk, which results in a tiny equity premium, even when we push risk aversion to very high values. Intuitively, households demand a return premium to hold equity investments because these investments will lose value in a bad state of the economy, precisely when households' consumption level falls and they would benefit from having more assets to sustain consumption. The more households' consumption could potentially fall (i.e. the more volatile their consumption is), the higher is the premium they demand to hold equity investments. In this class of model, consumption is too stable. With a small buffer of assets, households acquire a great capability of sustaining their consumption level in the face of negative shocks, much better than what is actually observed in the real economy. Such capability means that households won't charge much to hold risky assets, so the equity premium is small.

To test the effect of the consumption volatility on the equity premium, we simulated the two-asset model with high risk aversion, altering the aggregate productivity shock in order to double it's standard deviation. We do it, by calibrating the productivity in a low and high state of the economy as

$Z_l = 1.0405$ and $Z_h = 1.0820$, respectively. As result, we obtained a standard deviation of consumption growth of 3.4×10^{-3} and an equity premium of 0.09%, almost twice as large as the equity premium in the model with the baseline calibration for the productivity shock. Of course, this scenario implies an implausible level of fluctuation of the aggregate economy, as productivity might change by four percents from one quarter to another, but it serves to illustrate the role of volatility in generating a higher equity premium.

5.4 Additional Source of Heterogeneity

So far, our exposition was focused on measuring the amount of inequality generated by the portfolio mechanism. In this section, we introduce heterogeneity of preferences in order to bring the wealth inequality generated in the models close to the values observed in the data, and also to evaluate how the portfolio mechanism might interact with other sources of inequality.

We now assume that households' discount factor, β, is random and follows a Markov process, just like in Krusell and Smith (1998). Each household's β can take on three values, 0.09826, 0.09894 and 0.09962, and the transition probabilities are such that (i) the invariant distribution for β's has 80 percent of the population at the middle β and 10 percent at each of the other β's, (ii) immediate transitions between the extreme values of β do not occur, and (iii) the average duration of the highest and lowest β's is 50 years. For this simulation we set the parameter that controls risk aversion, γ, back to five and we change the inverse of the elasticity of substitution, ρ, to 1.4, because a higher EIS also helps to achieve a greater level of absolute inequality.

	Two-Asset Model	**One-Asset Model**
Wealth Gini	0.8210	0.7801
Equity/Debt ratio	53.0%	-
Equity premium (p.a.)	0.009%	-
Risk-free rate (p.a.)	2.837%	-

Table 5.3: Comparison between model economies, $\gamma = 5$, Beta heterogeneity

In this setting, as shown in table 5.3, the two-asset model is capable to reproduce the wealth inequality observed in the U.S. economy. Moreover, the difference of inequality between the models with and without portfolio choice gets amplified to 4.09 with the introduction of heterogeneity of preferences. This last result suggests the existence of an iteration between the portfolio mechanism and other sources of wealth inequality, which we discuss next.

The portfolio mechanism amplifies inequality as it provides households, who happened to accumulate more wealth, a superior expected return on their investments relative to that of less wealthier households. Therefore, features that increase the dispersion of households over the wealth dimension will have their effects enhanced by the portfolio mechanism. To further illustrate this notion, notice that increasing the dispersion of households over the wealth dimension, in general, means that the wealthier agents will reach even higher levels of wealth, so their superior expected return on savings will be applied to a greater base of assets. Meanwhile, a greater mass of households will remain at very low levels of wealth, receiving only the the minimum expected return on their savings (in case they have any). Hence, increasing the dispersion of households over wealth increases the inequality of financial income, which will further increase the inequality of wealth. This effect is stronger in the presence of the portfolio mechanism.

A direct consequence of this interaction, between the portfolio mechanism and other sources of inequality, is that it reduces the impact that these other sources must have in order to replicate the observed levels of wealth inequality. Therefore, a wider (and potentially more plausible) range of values could be used to calibrate these other process that generates inequality. For instance, had we successfully generated a sizable equity premium in the two-asset model, we would need a much lower dispersion of households' discount rate, β, in order to bring the wealth inequality of that model close to the values observed in the U.S. economy.

6
Conclusion

This work evaluates the impact, on wealth inequality, of introducing households' portfolio decisions in a heterogeneous agents model with aggregate uncertainty and a high level of risk aversion. Qualitatively, the model with portfolio choice is able to replicate the documented relation between households wealth level, the share of savings they allocate to equity investments, and the expected return they receive on savings. Also, wealth inequality do increase in the model with portfolio choice, showing that a portfolio mechanism operates in the direction of producing inequality.

Nonetheless, quantitatively, the model with portfolio decision fails to generate sizable wealth inequality when compared to a model where there is no portfolio decision and every household receives the same expected return on savings. The small effect of the portfolio mechanism on inequality occurs due to the lack of equity premium generated by the model.

We investigate why the levels of risk aversion used to reconcile the equity premium with the data on consumption of the U.S. economy fails to generate a high equity premium in the model with portfolio choice. We conclude that this class of model delivers a consumption growth volatility two orders of magnitude lower than the volatility that would be compatible with a risk premium high enough to replicate the equity premium. Lastly, we show there is a positive interaction between the portfolio mechanism and other processes that generate inequality.

Several models were developed in an attempt to replicate the equity premium. Most features of those models could be incorporated into the framework we presented in this paper. That would lead to a better quantitative fit of the equity premium and a better measure of the true contribution of the portfolio mechanism to generate wealth inequality. Incorporating those features, however, would require facing the challenge of implementing additional sources of heterogeneity in this already computationally demanding framework.

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